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ON L^1 CONVERGENCE RATE OF VISCOUS AND NUMERICAL APPROXIMATE SOLUTIONS OF GENUINELY NONLINEAR SCALAR CONSERVATION LAWS*

WEI-CHENG WANG†

Abstract. We study the rate of convergence of the viscous and numerical approximate solution to the entropy solution of genuinely nonlinear scalar conservation laws with piecewise smooth initial data. We show that the $O(\epsilon|\log\epsilon|)$ rate in L^1 is indeed optimal for viscous Burgers equation. Through the Hopf–Cole transformation, we can study the detailed structure of $\|u(\cdot,t)-u^\epsilon(\cdot,t)\|_{L^1}$. For centered rarefaction wave, the $O(\epsilon|\log\epsilon|)$ error occurs on the edges where the inviscid solution has a corner, and persists as long as the edges remain. The $O(\epsilon|\log\epsilon|)$ error must also occur at the critical time when a new shock forms automatically from the decreasing part of the initial data; thus it is, in general, impossible to maintain $O(\epsilon)$ rate for all $t>0$. In contrast to the centered rarefaction wave case, the $O(\epsilon|\log\epsilon|)$ error at critical time is transient. It resumes the $O(\epsilon)$ rate right after the critical time due to nonlinear effect. Similar examples of some monotone schemes, which admit a discrete version of the Hopf–Cole transformation, are also included.

Key words. hyperbolic conservation laws, error estimates, viscosity methods, monotone schemes

AMS subject classifications. 65M10, 65M05, 35L65

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1. Introduction.

The hyperbolic conservation law

$$\begin{aligned} &u_t + f(u)_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{1.1}$$

can be analyzed using the method of characteristics. Due to nonlinearity of f, the characteristic lines can intersect each other in finite time, and the solution develops jump discontinuities even if the initial data is smooth. Due to the presence of jump discontinuities, we need to generalize the solution class to include "weak solutions." In addition, since the weak solutions are not unique, entropy conditions are needed to specify physically meaningful weak solutions.

There are several equivalent forms of the entropy condition for genuinely nonlinear (say, $f'' > 0$) scalar conservation laws. Among them is the method of vanishing viscosity, which asserts that the physically relevant solution is obtained by solving the following viscous approximate equation

$$\begin{aligned} &u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx}, \quad x \in \mathbf{R}, \quad t > 0, \quad \epsilon > 0, \\ &u^\epsilon(x,0) = u_0(x) \end{aligned} \tag{1.2}$$

and letting ϵ go to zero. It is known that u^ϵ converges strongly, and the limiting function, u, is a weak solution of (1.1). Furthermore, u is the unique solution that satisfies the following entropy condition:

$$\frac{u(x+a,t)-u(x,t)}{a} \le \frac{E}{t}, \quad t > 0, \tag{1.3}$$

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†Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012. Current address: MSRI, 1000 Centennial Drive, Berkeley, CA 94720 (wangwc@msri.org).

where E is a constant depending only on the flux function f and initial data (see, for example, [11] for details).

Monotone difference schemes are first-order numerical schemes used to compute approximate solutions of (1.1):

$$\begin{aligned} w_j^{n+1} &= G(w_{j-p}^n, w_{j-p+1}^n, \ldots, w_{j+q}^n) \\ &= w_j^n - \lambda[\bar{f}(w_{j-p+1}^n, \ldots, w_{j+q}^n) - \bar{f}(w_{j-p}^n, \ldots, w_{j+q-1}^n)], \end{aligned} \tag{1.4}$$

where p and q are fixed nonnegative integers, G is a monotonely nondecreasing function in each of its arguments, $\bar{f}$ is a Lipchitz continuous function and is consistent with the scalar conservation law (1.1) in the sense

$$\bar{f}(w, \ldots, w) = f(w), \tag{1.5}$$

and $\lambda = \Delta t/\Delta x$ is a constant satisfying the CFL condition $\lambda < |f'|$.

Most well-known first-order schemes such as the Lax–Friedrichs scheme, Godunov scheme, and Enquist–Osher scheme are monotone schemes. Monotone schemes are known to converge to the entropy solution of (1.1) as $\Delta x \to 0$ (see [2]) and they are at most first-order accurate [1].

Whether or not a viscous approximation/monotone scheme can be of order $O(\epsilon)/O(\Delta x)$ accurate is an issue of practical interest and has long been studied. Although viscous approximation and monotone schemes are formally first order, they can really lose half-order accuracy across discontinuities. For example, it is easy to see, using a scaling argument, that the solution of the heat equation with an initial jump discontinuity is indeed $O(\sqrt{\epsilon})$ in L^1 norm away from its zero viscosity limit. In fact, Tang and Teng [14] proved that the $O(\sqrt{\epsilon})$ or $O(\sqrt{\Delta x})$ rate is indeed optimal for *all* monotone schemes applied to linear advection equations with discontinuous data.

For general BV initial data with genuinely nonlinear flux, several authors have obtained $O(\sqrt{\epsilon})$ or $O(\sqrt{\Delta x})$ rate. See, for example, Kuznetsov [6], Lucier [8], Sanders [10], and Tadmor [13]. It turns out to be optimal for this case (i.e., beyond linear degeneracy); see Sabac [9]. For the special case of monotonely nondecreasing initial data, Harabetian [5] has obtained $O(\epsilon|\log \epsilon|)/O(\Delta x|\log(\Delta x)|)$ rate in L^1 norm and showed that it is indeed optimal in this case.

Although BV solution is a natural class for genuinely nonlinear scalar conservation law, we will consider here only the subclass of piecewise smooth solutions with finitely many shocks. This class is of practical interest in shock capturing for the following reason: We expect viscous solution or monotone schemes to have better resolutions across an isolated jump discontinuity if the flux function is genuinely nonlinear, since for a linearly degenerate flux function, the discontinuity is a contact one, thus the smearing is a result of diffusion only; while in case of a genuinely nonlinear flux, the entropy condition dictates that the characteristic curves impinge into the shock, and thus tend to squeeze the profile in shape.

The first result in this direction was Goodman and Xin [4], where the authors considered piecewise smooth flows with noninteracting shocks for *systems* of hyperbolic conservation laws with viscosity. They obtained an $O(\epsilon)$ estimate away from shock regions and an overall $O(\epsilon^\gamma)$ rate for any $\gamma < 1$. The proof uses a matched asymptotic analysis employing a superposition of outer solutions (asymptotic series off the shock) and inner solutions (asymptotic expansion near the shock in stretched variables), as well as a nonlinear stability analysis based on energy estimates.

Inspired by [4], Teng and Zhang [16], Tang and Teng [15] showed that, for genuinely nonlinear scalar conservation laws with piecewise smooth initial data having finitely many inflection points, the convergence rate can be improved to $\sup_{t>0} \|u(\cdot,t) - u^\epsilon(\cdot,t)\| \le O(\epsilon|\log\epsilon|)$. And in case there is no centered rarefaction wave or shock formation at a later time, the rate is actually $O(\epsilon)$.

The corresponding counterpart for monotone schemes is more subtle. Engquist and Yu [3] and Smyrlis and Yu [12] obtained pointwise estimates for a wide class of finite difference schemes, which result in the $O(\Delta x)$ convergence rate in L^1 norm of monotone schemes for piecewise smooth initial data with noninteracting shocks provided no shocks form at a later time.

From the argument in [15], it is not clear whether the $O(\epsilon|\log\epsilon|)$ is optimal beyond the centered rarefaction wave case (the optimality of the centered rarefaction wave case was shown in Harabetian [5]). In this paper, we will study in detail the structure of $\|u(\cdot,t) - u^\epsilon(\cdot,t)\|$ through an example, the viscous Burgers equation, to gain more insight. It turns out that this rate is actually obtained at the critical time when a shock develops from the decreasing part of the initial data. Thus, it is in general impossible to maintain $O(\epsilon)/O(\Delta x)$ rate for all $t > 0$. However, in contrast to the centered rarefaction wave case, this phenomenon is transient; it resumes the $O(\epsilon)/O(\Delta x)$ rate right after the critical time. (This case was not covered in [14], [15], [3], and [12], where the authors considered the shocks coming from jumps in initial data.) This result is consistent with the following heuristic argument: The viscous approximation/monotone schemes are first-order accurate both before and after the critical time, but for different reasons. Before the critical time, the solution of (1.1) is smooth if the initial data is; therefore, the viscous term ϵu^ϵ_{xx} is an $O(\epsilon)\cdot O(1)$ quantity. After the critical time, the shock is already formed, and the impinging of characteristic lines counteract with diffusion. However, at the critical time, neither of these mechanisms is available, resulting in an underresolution.

The rest of this paper is arranged as follows: In section 2, we will review some basic facts about formation of shocks, the Hopf–Cole transformation, and a few lemmas to be used later. In section 3.1, we state and prove the main theorem concerning the convergence rate at and after critical time for the viscous Burgers equation using the Hopf–Cole transformation. In section 3.2, we give the same results for several monotone schemes with particular flux functions, including upwind, Lax–Friedrichs, and Godunov scheme, which admit a discrete version of Hopf–Cole transformation. It will be clear how these elementary arguments can be utilized to study the centered rarefaction wave case, and interactions of shocks and centered rarefaction waves, etc. The results are as stated in the abstract of this paper; we thus omit the details.

2. Preliminaries.

Notation: $\|\cdot\|$ is the L^1 norm. We'll also denote the local L^1 integral $\int_a^b |g(x)|dx$ by $\|g\|_{L^1(dx;[a,b])}$.

Consider the viscous and inviscid Burgers equation which are special cases of (1.1) and (1.2) with $f(u) = \frac{1}{2}u^2$,

$$\begin{aligned} &u_t + uu_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &u^\epsilon_t + u^\epsilon u^\epsilon_x = \epsilon u^\epsilon_{xx}, \quad x \in \mathbf{R}, \quad t > 0, \quad \epsilon > 0, \\ &u^\epsilon(x,0) = u_0(x). \end{aligned} \tag{2.2}$$

We first recall some facts about spontaneous formation of shocks. If the initial data is smooth and is such that $f'(u_0(\cdot))$ is not monotonely nondecreasing, the characteristic lines can intersect each other and the shock forms. If $\xi_1 < \xi_2$ with $f'(u_0(\xi_1)) > f'(u_0(\xi_2))$, then the two characteristic lines starting from ξ_1 and ξ_2 intersect at time $t = \frac{\xi_2 - \xi_1}{f'(u_0(\xi_1)) - f'(u_0(\xi_2))}$; thus the first time at which neighboring characteristic lines intersect is when $t = t_c = \frac{-1}{\min_\xi \frac{d}{d\xi} f'(u_0(\xi))}$ and the initial shock is located at the characteristic line starting from ξ_0 where the minimum is taken.

Here in Burgers's equation, $f'(u_0) = u_0$. Up to a Galilean transformation, we may assume that $u_0(0) = 0$ and that $\xi = 0$ is where u_0' assumes its negative minimum which corresponds to the initial formation of the shock. Thus the local Taylor expansion near $\xi = 0$ reads

$$u_0(\xi) = -\frac{1}{t_c}\xi + a\xi^{2p+1} + \cdots, \tag{2.3}$$

where $a > 0$ and p is a positive integer. We'll carry out the analysis for $p = 1$; the proof for other values of p is similar.

By differentiating (2.1) with respect to x and then integrating along the characteristic lines, one can find that the derivative blows up near $t = t_c$ (for a detail derivation, see, for example, [11]),

$$u_x(0,t) = -\frac{O(1)}{t_c - t}, \ t < t_c. \tag{2.4}$$

Our main tool is the classical Hopf–Cole transformation,

$$u^\epsilon = -2\epsilon(\log\phi^\epsilon)_x. \tag{2.5}$$

Through (2.5), (2.2) linearizes to the heat equation

$$\phi^\epsilon_t = \epsilon\phi^\epsilon_{xx}. \tag{2.6}$$

After transforming the initial data and solving the heat equation, we have

$$u^\epsilon(x,t) = -2\epsilon\left(\log\int_{-\infty}^{\infty} e^{-\frac{1}{2\epsilon}G(x,y,t)}dy\right)_x, \tag{2.7}$$

where $G(x,y,t) = \int_0^y u_0(y')dy' + \frac{(x-y)^2}{2t}$.

Since (2.7) gives an exact formula for $u^\epsilon(x,t)$, hence for $\int^x u^\epsilon(x',t)dx'$, we can estimate $\|u^\epsilon(\cdot,t) - u(\cdot,t)\|$ as long as we know the sign of $u^\epsilon(\cdot,t) - u(\cdot,t)$. The following lemma is based on this observation.

LEMMA 2.1. *Let u_0 be a smooth and bounded function satisfying*

(A.1) $u_0(\xi) = -\frac{\xi}{t_c} + a\xi^3 + b\xi^4 + O(\xi^5)$ *for* $|\xi| < \delta$ *where* $a > 0$*;*

(A.2) $\xi = 0$ *is the point corresponding to the first spontaneous formation of shocks, that is,* $u_0'(\xi) > -\frac{1}{t_c}$ *for all* $\xi \neq 0$*;*

(A.3) u_0 *is antisymmetric:* $u_0(-\xi) = -u_0(\xi)$*;*

(A.4) u_0 *is monotonely decreasing;*

(A.5) u_0 *is concave on* $\xi < 0$*, and, therefore, by Assumption* (A.3)*, convex on* $\xi > 0$.

Then

$$u(x,t_c) \geq (\leq) u^\epsilon(x,t_c) \quad \text{on } x < (>)0. \tag{2.8}$$

Proof. By symmetry, we only need to prove the statement on $\{x < 0\}$. We will apply the maximum principle in the region $\{(x,t) : 0 < t < t_c, x < 0\}$ for $w = u^\epsilon - u$, which satisfies

$$w_t + (aw)_x - \epsilon w_{xx} = \epsilon u_{xx}, \tag{2.9}$$

where $a(x,t) = \frac{1}{2}(u^\epsilon(x,t) + u(x,t)) = \frac{1}{2}w(x,t) + u(x,t)$. Clearly, $w = 0$ on $\{(x,t) : t = 0, x < 0\}$ by definition and on $\{(x,t) : 0 < t < t_c, x = 0\}$ by symmetry. Since monotonicity and concavity of u is preserved under the characteristic flow (one can see this by differentiating (1.1) once and twice, then integrating along the characteristic lines), we have the correct signs on the right-hand side of (2.8) and the coefficient of w in order to apply the maximum principle, by which we conclude that $w \le 0$ on $\{x < 0\}$. ∎

We'll also need the following lemma.

LEMMA 2.2 (L^1 stability). *If $u_i^\epsilon(x,t)$, $i = 1, 2$ satisfy*

$$\frac{\partial}{\partial t}u_i^\epsilon + \frac{\partial}{\partial x}f(u_i^\epsilon) - \epsilon\frac{\partial^2}{\partial x^2}u_i^\epsilon = g_i(x,t), \tag{2.10}$$

then

$$\|u_1^\epsilon(\cdot,t) - u_2^\epsilon(\cdot,t)\| \le \|u_1^\epsilon(\cdot,0) - u_2^\epsilon(\cdot,0)\| + \int_0^t \|g_1(\cdot,s) - g_2(\cdot,s)\|ds. \tag{2.11}$$

Proof. Let $w = u_1^\epsilon - u_2^\epsilon$; then w solves the following equation:

$$w_t + (aw)_x - \epsilon w_{xx} = g_1 - g_2, \tag{2.12}$$

where $a(x,t) = \frac{1}{2}(u_1^\epsilon + u_2^\epsilon)$ for Burgers's equation. (For general flux, $a(x,t)$ is a proper average of $f'(u_1^\epsilon(x,t))$ and $f'(u_2^\epsilon(x,t))$.) Since the backward adjoint equation

$$\begin{aligned} &z_t + az_x + \epsilon z_{xx} = 0, \\ &z(\cdot,t) = \operatorname{sgn}(w(\cdot,t)) \end{aligned} \tag{2.13}$$

satisfies the maximum principle, we then complete the proof by integrating $z \cdot (2.12) + w \cdot (2.13)$ by parts. ∎

3. Convergence rate at and near critical time.

3.1. The Burgers equation.

THEOREM 3.1. *Let $u^\epsilon(x,t)$ and $u(x,t)$ be solutions of* (2.1) *and* (2.2)*, respectively, with the same initial data $u_0(x)$ satisfying* (A.1) *and* (A.2) *in Lemma* 2.1*, then for t near t_c, we have*

1. *If $t \ne t_c$, then*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\| \le C(t)\epsilon \quad \text{as} \quad \epsilon \to 0,$$

where $C(t) = O(\log \frac{1}{|t-t_c|})$.

2.

$$\|u^\epsilon(\cdot,t_c) - u(\cdot,t_c)\| = O(\epsilon|\log\epsilon|) \quad \text{as} \quad \epsilon \to 0.$$

Proof. The case $t < t_c$ of the first part is a direct consequence of Lemma 2.2 above, since $\|u_{xx}(\cdot,t)\| = TV(u_x(\cdot,t)) = 2\|u_x(\cdot,t)\|_{L^\infty} = -2u_x(0,t) = O(\frac{1}{t_c-t})$.

At $t = t_c$, we first prove the special case where the initial data satisfy the assumptions of Lemma 2.1. In this case we see that from (2.8),

$$\|u^\epsilon(\cdot,t_c) - u(\cdot,t_c)\|_{L^1(dx;[-1,0])} = \int_{-1}^0 u(x,t_c)dx - \int_{-1}^0 u^\epsilon(x,t_c)dx. \tag{3.1.1}$$

By the Hopf–Cole transformation,

$$\int_{-1}^{0} u^\epsilon(x,t)dx = -2\epsilon \log\left(\frac{\displaystyle\int e^{-\frac{1}{2\epsilon}G(0,y,t)}dy}{\displaystyle\int e^{-\frac{1}{2\epsilon}G(-1,y,t)}dy}\right), \tag{3.1.2}$$

where $G(x,y,t) = \int_0^y u_0(y')dy' + \frac{(x-y)^2}{2t}$ and the domain of integration in $int(\cdot)dy$ is the whole real line. Following the standard stationary phase method, we check that $G_{yy}(-1,\xi(-1,t_c),t_c) = u_0'(\xi) + \frac{1}{t_c} > 0$, where $\xi = \xi(x,t)$ is where $G(x,\cdot,t)$ assumes its global minimum,

$$u_0(\xi(x,t)) = \frac{x-\xi(x,t)}{t}. \tag{3.1.3}$$

Thus at $t = t_c$, the leading-order asymptotic expansion of the denominator in (3.1.2) is

$$\begin{aligned}\int e^{-\frac{1}{2\epsilon}G(-1,y,t_c)}dy &= e^{-\frac{1}{2\epsilon}G(-1,\xi(-1,t_c),t_c)}\int e^{-\frac{1}{2\epsilon}[G(-1,y,t_c)-G(-1,\xi(-1,t_c),t_c)]}dy\\ &\sim e^{-\frac{1}{2\epsilon}G(-1,\xi(-1,t_c),t_c)}\int e^{-\frac{1}{2\epsilon}\frac{G_{yy}(-1,\xi(-1,t_c),t_c)}{2}(y-\xi(-1,t_c))^2}dy\\ &= \frac{2\sqrt{\pi}}{u_0'(\xi(-1,t_c))+\frac{1}{t_c}}\cdot\epsilon^{\frac{1}{2}}\exp\left(-\frac{1}{2\epsilon}G(-1,\xi(-1,t_c),t_c)\right).\end{aligned} \tag{3.1.4}$$

The numerator, however, has a quartic exponent $G(0,y,t_c)$ at $(x,t) = (0,t_c)$, since $\xi(0,t_c) = 0$, $G_y(0,\xi(0,t_c),t_c) = G_{yy}(0,\xi(0,t_c),t_c) = G_{yyy}(0,\xi(0,t_c),t_c) = 0$ and $G_{yyyy}(0,\xi(0,t_c),t_c) = 6a > 0$. Thus the asymptotic expansion of the integral is, to leading order,

$$\begin{aligned}\int e^{-\frac{1}{2\epsilon}G(0,y,t_c)}dy &= e^{-\frac{1}{2\epsilon}G(0,0,t_c)}\int e^{-\frac{1}{2\epsilon}[G(0,y,t_c)-G(0,0,t_c)]}dy\\ &\sim e^{-\frac{1}{2\epsilon}G(0,0,t_c)}\int e^{-\frac{1}{8\epsilon}ay^4}dy\\ &= I_0(\frac{4\epsilon}{a})^{\frac{1}{4}}\exp\left(-\frac{1}{2\epsilon}G(0,0,t_c)\right),\end{aligned} \tag{3.1.5}$$

where $I_0 = \int_{-\infty}^{\infty} e^{-\frac{z^4}{2}}dz$ is a constant. Therefore,

$$\int_{-1}^{0} u^\epsilon(x,t_c)dx \sim G(0,0,t_c) - G(-1,\xi(-1,t_c),t_c) + \frac{1}{2}\epsilon\log\epsilon + \cdots. \tag{3.1.6}$$

By differentiating (3.1.3) with respect to x, we see that $\frac{\partial}{\partial x}G(x,\xi(x,t),t) = u(x,t)$, so

$$G(0,0,t_c) - G(-1,\xi(-1,t_c),t_c) = \int_{-1}^{0} u(x,t_c)dx. \tag{3.1.7}$$

From (3.1.2), (3.1.4), (3.1.5), (3.1.6), and (3.1.7), we conclude that

$$\|u^\epsilon(\cdot,t_c) - u(\cdot,t_c)\|_{L^1(dx;[-1,0])} \sim \frac{1}{2}\epsilon|\log\epsilon|. \tag{3.1.8}$$

The same estimate holds for $\|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[0,1])}$ by symmetry. The integral outside of $[-1,1]$ is of lower order by virtue of Lemma 3.2 below. Thus the special case is proved.

To prove the general case, we note that, because of the structure of the initial data, the assumptions of Lemma 2.1, except (A.3), indeed hold for ξ near zero in general. Thus we only have to take care of the antisymmetry. We proceed as follows.

Let $\delta_0>0$ be a small number such that all the assumptions in Lemma 2.1, except (A.3), are valid for $|\xi|<2\delta_0$, and let the characteristic line starting from $(-\delta_0,0)$ intersect the line $\{t=t_c\}$ at $(-\delta_1,t_c)$. We will concentrate on the local deviation $\|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])}$. The following lemma allows us to modify the initial data in order to reduce to the antisymmetric case.

LEMMA 3.2. *Let v_0, w_0 be two bounded initial data such that $v_0(\xi)=w_0(\xi)$ on $\{\xi\geq\alpha\}$ for some $\alpha\in\mathbf{R}$ and let $v^\epsilon(x,t)$, $w^\epsilon(x,t)$ be corresponding solutions of the viscous Burgers equation. If for some $\beta>\alpha$, the characteristic flows of v_0 and w_0 left of α are strictly separated from those right of β up to some time $t_1>0$, that is, if there exist $\beta_1>\alpha_1$, such that the characteristic lines of v_0 and w_0 starting from left of $(\alpha,0)$ intersect the line $\{t=t_1\}$ at left of (α_1,t_1), and vice versa on the right of $(\beta,0)$ and (β_1,t_1), then*

$$\|v^\epsilon(\cdot,t_1)-w^\epsilon(\cdot,t_1)\|_{L^1(dx;[\beta_1,\infty))}=O(1)e^{-\frac{O(1)}{\epsilon}}. \tag{3.1.9}$$

Proof. Equation (3.1.9) can be proved by estimating the Green function of the backward adjoint equation, or one can prove it directly using the Hopf–Cole transformation. ∎

Therefore, by adjusting $u_0(\xi)$ on $\{\xi<-2\delta_0\}$ if necessary, we may assume that u_0 satisfies the assumptions (A.2), (A.4), and (A.5) globally on $\{\xi<0\}$. To reduce to the special case, we now construct an antisymmetric initial data

$$u_{a,0}(\xi)=\begin{cases} u_0(\xi) & \text{if } \xi<0,\\ -u_0(-\xi) & \text{if } \xi\geq 0.\end{cases} \tag{3.1.10}$$

Since the corresponding inviscid solutions agree on the interval under consideration,

$$u_a(x,t_c)=u(x,t_c) \quad \text{on} \quad -\delta_1\leq x\leq 0, \tag{3.1.11}$$

it suffices to estimate $\|u_a^\epsilon(\cdot,t_c)-u^\epsilon(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])}$.

By Assumption (A.1), we have $u_{a,0}(\xi)-u_0(\xi)\leq(\geq)0$ on $\{\xi\leq 2\delta_0\}$ if $b>(<)0$. By Lemma 3.2, we can adjust $u_0(\xi)$ on $\{\xi>2\delta_0\}$ if necessary, so we may assume, without loss of generality, that

$$u_{a,0}(\xi)-u_0(\xi)\leq(\geq)0 \ \text{ for all } \ \xi\in\mathbf{R} \ \text{ if } \ b>(<)0. \tag{3.1.12}$$

From the classical comparison lemma [2], (3.1.12) implies that

$$u_a^\epsilon(x,t)-u^\epsilon(x,t)\leq(\geq)0 \ \text{ if } \ b>(<)0, \tag{3.1.13}$$

and, therefore,

$$\|u_a^\epsilon(\cdot,t_c)-u^\epsilon(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])}=\left|\int_{-\delta_1}^0 u_a^\epsilon(x,t_c)dx-\int_{-\delta_1}^0 u^\epsilon(x,t_c)dx\right| \tag{3.1.14}$$

Equations (3.1.11), (3.1.13), and (3.1.14) together imply

$$\|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])},$$
$$(3.1.15)\quad = \begin{cases} \displaystyle\int_{-\delta_1}^0 u(x,t_c)dx - \int_{-\delta_1}^0 u^\epsilon(x,t_c)dx & \text{if } b<0,\\ \displaystyle\int_{-\delta_1}^0 u(x,t_c)dx + \int_{-\delta_1}^0 u^\epsilon(x,t_c)dx - 2\int_{-\delta_1}^0 u_a^\epsilon(x,t_c)dx & \text{if } b>0; \end{cases}$$

therefore, we can apply the Hopf–Cole transformation again,

$$(3.1.16)\quad \|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])},$$
$$= \begin{cases} -2\epsilon\log\left(\dfrac{\displaystyle\int e^{-\frac{1}{2\epsilon}G(-\delta_1,y,t_c)-G(-\delta_1,\xi(-\delta_1,t_c),t_c)}dy}{\displaystyle\int e^{-\frac{1}{2\epsilon}G(0,y,t_c)-G(0,0,t_c)}dy}\right) & \text{if } b<0,\\ -2\epsilon\left[\log\left(\dfrac{\displaystyle\int e^{-\frac{1}{2\epsilon}G_a(-\delta_1,y,t_c)-G_a(-\delta_1,\xi(-\delta_1,t_c),t_c)}dy}{\displaystyle\int e^{-\frac{1}{2\epsilon}G_a(0,y,t_c)-G_a(0,0,t_c)}dy}\right)\right. & \\ \left.+\log\left(\dfrac{\displaystyle\int e^{-\frac{1}{2\epsilon}G(0,y,t_c)}dy}{\displaystyle\int e^{-\frac{1}{2\epsilon}G_a(0,y,t_c)}dy}\right)\right] & \text{if } b>0, \end{cases}$$

where $G_a(x,y,t) = \int_0^y u_{a,0}(y')dy' + \frac{(x-y)^2}{2t}$, and $G_a(-\delta_1,\xi(-\delta_1,t_c),t_c) = G(-\delta_1,\xi(-\delta_1,t_c),t_c)$ cancel out in the second term of the case $b>0$ in (3.1.17). Since $u_0'''(0)$ is preserved under antisymmetrization, we see that from (3.1.5)

$$(3.1.17)\quad \frac{\displaystyle\int e^{-\frac{G(0,y,t_c)}{2\epsilon}}dy}{\displaystyle\int e^{-\frac{G_a(0,y,t_c)}{2\epsilon}}dy} = 1+o(1).$$

In view of (3.1.17) and (3.1.17), we have

$$(3.1.18)\quad \|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[-\delta_1,0])} \sim \frac{1}{2}\epsilon|\log\epsilon|.$$

The estimate for $\|u^\epsilon(\cdot,t_c)-u(\cdot,t_c)\|_{L^1(dx;[0,\delta_1])}$ is similar. The general case for $t=t_c$ is thus proved.

The case $t>t_c$ can be reduced to the case $t<t_c$ by constructing a new initial data which delays the formation of the shock. Let $t_0>t_c$ be given, with t_0-t_c sufficiently small. Denote by $s(t)$ the location of the shock at time t, and let $(\xi_-,0)$ be where the backward characteristic line from $(s(t_0)-0,t_0)$ intersects the x-axis. For t_0-t_c sufficiently small, ξ_- is close to 0 and the tangent line of $u_0(\cdot)$ at $(\xi_-,u_0(\xi_-))$ lies above u_0 in a neighborhood of ξ_-. Now define

$$\bar u_0(\xi) = \begin{cases} u_0(\xi) & \text{if } \xi<\xi_-,\\ \max(u_0(\xi), u_0(\xi)+u_0'(\xi_-)(\xi-\xi_-)\,) & \text{if } \xi\ge\xi_-, \end{cases}$$

and let $\bar u(x,t)$ and $\bar u^\epsilon(x,t)$ be correponding inviscid and viscous solutions. It is easy to see that

(a) $\bar{u}(x,t_0) = u(x,t_0)$ for $x < s(t_0)$.
(b) The critical time for $\bar{u}_0$ is $\bar{t}_c = -\frac{1}{u_0'(\xi_-)} > t_0$; thus no shock forms in $\bar{u}(\cdot,\cdot)$ up to $t = t_0$. Moreover, $\bar{t}_c - t_0 = \frac{1}{2}(t_0 - t_c) + O((t_0 - t_c)^2)$.
(c) $\bar{u}_0(\xi) \geq u_0(\xi)$ and thus $\bar{u}^\epsilon(x,t) \geq u^\epsilon(x,t)$.

From (b), we have

$$\|\bar{u}^\epsilon(\cdot,t_0) - \bar{u}(\cdot,t_0)\| = O\left(\log \frac{1}{|t-t_c|}\right)\epsilon \quad \text{as} \quad \epsilon \to 0, \tag{3.1.19}$$

and from (c)

$$\begin{aligned} &\|u^\epsilon(\cdot,t_0) - \bar{u}^\epsilon(\cdot,t_0)\|_{L^1(dx;[-1,s(t_0)])} \\ &= \int_{-1}^{s(t_0)} \bar{u}^\epsilon(x,t_0)dx - \int_{-1}^{s(t_0)} u^\epsilon(x,t_0)dx \\ &= -2\epsilon\left[\log\left(\frac{\int e^{-\frac{1}{2\epsilon}\bar{G}(s(t_0),y,t_0)}dy}{\int e^{-\frac{1}{2\epsilon}G(s(t_0),y,t_0)}dy}\right) - \log\left(\frac{\int e^{-\frac{1}{2\epsilon}\bar{G}(-1,y,t_0)}dy}{\int e^{-\frac{1}{2\epsilon}G(-1,y,t_0)}dy}\right)\right], \end{aligned} \tag{3.1.20}$$

where $\bar{G}(x,y,t) = \int_0^y \bar{u}_0(y')dy' + \frac{(x-y)^2}{2t}$. A standard process of asymptotic expansion leads to

$$\frac{\int e^{-\frac{1}{2\epsilon}\bar{G}(-1,y,t_0)}dy}{\int e^{-\frac{1}{2\epsilon}G(-1,y,t_0)}dy} \sim 1 + O(\epsilon). \tag{3.1.21}$$

As to the first term in (3.1.20), we note that the exponent $G(s(t_0),\cdot,t_0)$ indeed has two global minima occurring at ξ_- and ξ_+ due to the presence of the shock. Here ξ_+ is where the backward characteristic line from $(s(t_0)+0,t_0)$ intersects the x-axis. Since $\bar{u}(\cdot,t_0)$ is smooth, there is only one global minimum of $\bar{G}(s(t_0),\cdot,t_0)$ occurring at ξ_-, therefore,

$$\frac{\int e^{-\frac{1}{2\epsilon}\bar{G}(s(t_0),y,t_0)}dy}{\int e^{-\frac{1}{2\epsilon}G(s(t_0),y,t_0)}dy} \sim \frac{(u_0'(\xi_-)+\frac{1}{t_0})^{\frac{1}{2}}}{(u_0'(\xi_-)+\frac{1}{t_0})^{\frac{1}{2}} + (u_0'(\xi_+)+\frac{1}{t_0})^{\frac{1}{2}}} + o(1) < 1. \tag{3.1.22}$$

From (a), (3.1.19), (3.1.20), (3.1.21), and (3.1.22), we conclude that

$$\|u^\epsilon(\cdot,t_0) - u(\cdot,t_0)\|_{L^1(dx;[-1,s(t_0)])} = O\left(\log\frac{1}{|t-t_c|}\right)\epsilon. \tag{3.1.23}$$

A similar estimate holds for $\|u^\epsilon(\cdot,t_0) - u(\cdot,t_0)\|_{L^1(dx;[s(t_0),1])}$, and the theorem is proved. □

Remark 1. It is clear from the proof that the $O(\epsilon|\log\epsilon|)$ rate is indeed optimal at the critical time. For a general exponent $2p+1$ in (2.3), the constant $\frac{1}{2}$ in (3.1.8) is replaced by $\frac{p}{p+1}$.

The idea used in the proof of Lemma 2.1 and Theorem 3.1 can be carried over to analyze the structure of the error in the case of a centered rarefaction wave. The

$O(\epsilon|\log\epsilon|)$ error is optimal and, roughly speaking, is restricted to the inner edges of the fan.

PROPOSITION 3.3. *Let the initial data $u_0(\xi)$ be a piecewise smooth function with a jump discontinuity at $\xi = 0$ and $u_0(0^-) < u_0(0^+)$ but smooth otherwise. Assume further that u_0 is concave on $\{\xi > 0\}$, convex on $\{\xi < 0\}$, and monotonely nondecreasing. Then the following local L^1 error estimates holds. For c, $d \in R$ satisfying*

$$\frac{u_0(0^+) + u_0(0^-)}{2} t < c < u_0(0^+)t < d,$$

$$\begin{array}{lcl} \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[\frac{u_0(0^+)+u_0(0^-)}{2}t,c])} & \sim & C_1\epsilon, \\ \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[c,u_0(0^+)t])} & \sim & \epsilon|\log\epsilon|, \\ \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[u_0(0^+)t,d])} & \sim & C_2\epsilon, \end{array} \tag{3.1.24}$$

where

$$C_1 = 2\log\left(\frac{\frac{1}{\frac{c}{t}-u_0(0^-)} + \frac{1}{u_0(0^+)-\frac{c}{t}}}{\frac{4}{u_0(0^+)-u_0(0^-)}}\right), \quad C_2 = 2\log\left(2\sqrt{\frac{u_0(\xi(u_0(0^+)t,t))}{u_0(\xi(d,t))}}\right), \tag{3.1.25}$$

and $\xi(x,t)$ is defined implicitly by (3.1.3). *Similar estimates hold for intervals at left of the center $\frac{u_0(0_+)+u_0(0_-)}{2}t$.*

The proof is similar to the proof of Theorem 3.1; we omit the detail.

We remark here that the monotonicity and concavity (convexity) assumptions in Proposition 3.3 are not essential; one can treat the case of a general centered rarefaction wave up to the time when the edge is, if ever, merged into a shock. The estimates (3.1.24) remain valid except the constants C_1 and C_2 may become larger due to overestimates. After the edge is merged into a shock, the local L^1 error reduces to $O(\epsilon)$.

The precise form of the statement above is rather complicated; we illustrate with the following example instead.

Example. Consider (2.1) and (2.2) with initial data

$$u_0(\xi) = \begin{cases} -1, & \xi < 0, \\ 1 - \frac{\xi}{2}, & 0 \le \xi < 1, \\ -\frac{\xi}{2}, & 1 \le \xi < 2, \\ -1, & 2 \le \xi. \end{cases} \tag{3.1.26}$$

At time $t < 1$, the solution to (1.1) has a rarefaction wave spanning over $-t \le x \le t$ and a standing shock at $\xi = 1$. At $t = 1$, the right edge of the centered rarefaction wave is confronted with the standing shock and merged into it afterward. The following local L^1 estimates hold.

For $0 < t < 1$, we have

$$\begin{array}{lcl} \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[0,t])} & \sim & \epsilon|\log\epsilon|, \\ \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[t,1])} & \sim & O(\epsilon), \\ \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[1,\infty))} & \sim & O(\epsilon). \end{array} \tag{3.1.27}$$

At $t = 1$,

$$\begin{array}{lcl} \|u(\cdot,1) - u^\epsilon(\cdot,1)\|_{L^1(dx;[0,1])} & \sim & \epsilon|\log\epsilon|, \\ \|u(\cdot,1) - u^\epsilon(\cdot,1)\|_{L^1(dx;[1,\infty))} & \sim & O(\epsilon). \end{array} \tag{3.1.28}$$

After the interaction, say, $1 < t < 1.5$, the shock begins to move. Denoting by $s(t)$ the shock location, we have

$$\begin{array}{rcl} \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[0,s(t)])} & \sim & O(\epsilon), \\ \|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[s(t),\infty))} & \sim & O(\epsilon). \end{array} \tag{3.1.29}$$

We outline the computation for the first equation in (3.1.27); the rest is done in a similar way. Consider

$$\bar{u}_0(\xi) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0, \end{cases} \tag{3.1.30}$$

and denote the corresponding viscous and inviscid solutions by $\bar{u}^\epsilon$ and $\bar{u}$, respectively. By a variant of Lemma 2.1, we can conclude that $\bar{u}(x,t) \geq (\leq) \bar{u}^\epsilon(x,t)$ on $\{x > 0\}$ ($\{x < 0\}$). Thus one can apply the Hopf–Cole transformaton. A short calculation leads to $\bar{G}(x,y,t) = \int_0^y \bar{u}_0(y')dy' + \frac{(x-y)^2}{2t}$ near the absolute minimum,

$$\begin{array}{rcl} \bar{G}(0,y,t) & \sim & |y|, \text{ for } y \text{ near } 0, \\ \bar{G}(t,y,t) & \sim & \left\{ \begin{array}{l} \frac{t}{2} - 2y, \ y < 0 \\ \frac{t}{2} + \frac{y^2}{2t}, \ y > 0 \end{array} \right. \quad \text{for } y \text{ near } 0, \end{array} \tag{3.1.31}$$

from which one easily concludes that

$$\|\bar{u}(\cdot,t) - \bar{u}^\epsilon(\cdot,t)\|_{L^1(dx;[0,t])} \sim \epsilon|\log\epsilon|. \tag{3.1.32}$$

On the other hand, $\bar{u}_0 \geq u_0$, thus $\bar{u}^\epsilon \geq u^\epsilon$ and we can apply the Hopf–Cole transformation again. The same calculation leads to

$$\begin{array}{rcl} G(0,y,t) & \sim & |y|, \text{ for } y \text{ near } 0, \\ G(t,y,t) & \sim & \left\{ \begin{array}{l} \frac{t}{2} - 2y, \ y < 0[1ex] \\ \frac{t}{2} + (\frac{1}{2t} - \frac{1}{4})y^2, \ y > 0 \end{array} \right. \quad \text{for } y \text{ near } 0; \end{array} \tag{3.1.33}$$

one thus concludes that

$$\|\bar{u}^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(dx;[0,t])} \sim \epsilon \log\left(\frac{2-t}{2}\right). \tag{3.1.34}$$

We conclude with the first equation of (3.1.27), with the triangle inequality and the fact that $u(x,t)$ coincides with $\bar{u}(x,t)$ for $x \leq t$.

3.2. Hopf–Cole–Lax transformation for some monotone schemes. We now give another example in which the convergence rate is not first order at the critical time—Lax–Friedrichs scheme applied to the conservation law (1.1) with a specific flux function:

$$f_L(u) = \log\left(\frac{\cosh(u)+1}{2}\right).$$

The Lax–Friedrichs scheme for this particular flux admits a discrete version of Hopf–Cole transformation. This was first observed by Lax [7] for upwind scheme with a family of flux function $f(u) = \log(a + be^{-u})$, $a, b > 0$, $a + b = 1$. Here we adopt a variation of the original one in order to maintain symmetry, which simplifies the analysis.

The following properties of $f_L(u)$ are elementary:

(C.1) $f_L(u) = f_L(-u)$.
(C.2) $f_L(0) = 0$.
(C.3) $f'_L(u) = \frac{\sinh(u)}{\cosh(u)+1}$, $f'_L(0) = 0$.
(C.4) $f''_L(u) = \frac{1}{\cosh(u)+1} > 0$.

We now study the convergence rate for the Lax–Friedrichs scheme with f_L. Let $u^\Delta(x,t)$ be the approximate solution obtained via the Lax–Friedrichs scheme

$$u^\Delta(x,t+\Delta) = \frac{1}{2}(\ u^\Delta(x-\Delta,t)+u^\Delta(x+\Delta,t)\ )-\frac{1}{2}(\ f_L(u^\Delta(x+\Delta,t))-f_L(u^\Delta(x-\Delta,t))\ ), \tag{3.2.1}$$

where the argument (x,t) is restricted to grid points only, and we have put $\Delta x = \Delta t = \Delta$ for simplicity.

Now let $U^\Delta(x,t) = 2\Delta \sum_{k=-\infty^0} u^\Delta(x-2k\Delta,t)$. The equation for U^Δ is

$$U^\Delta(x,t+\Delta) = \frac{1}{2}(\ U^\Delta(x-\Delta,t)+U^\Delta(x+\Delta,t)\ )-\Delta f_L\left(\frac{U^\Delta(x+\Delta,t)-U^\Delta(x-\Delta,t)}{2\Delta}\right). \tag{3.2.2}$$

Now we apply the Hopf–Cole–Lax transformation

$$U^\Delta = G(V^\Delta) = -2\Delta\log(V^\Delta),$$

which brings (3.2.1) to

$$\begin{aligned} G(V^\Delta(x,t+\Delta)) = \quad & \frac{1}{2}[G(V^\Delta(x+\Delta,t))+G(V^\Delta(x-\Delta,t))][1ex] \\ & -\Delta f_L\left(\frac{G(V^\Delta(x+\Delta,t))-G(V^\Delta(x-\Delta,t))}{2\Delta}\right). \end{aligned} \tag{3.2.3}$$

The equation for V^Δ thus linearizes as the following identity holds for all V, $W \in R$,

$$\frac{1}{2}(G(V)+G(W)) - \Delta f_L\left(\frac{G(V)-G(W)}{2\Delta}\right) = G\left(\frac{V+W}{2}\right).$$

Thus

$$V^\Delta(x,t+\Delta) = \frac{1}{2}(\ V^\Delta(x+\Delta,t)+V^\Delta(x-\Delta,t)\ ),$$

and, therefore,

$$V^\Delta(x,t) = \sum_{l=0}^{n}\binom{n}{l}\frac{1}{2^n}V^\Delta(x-n\Delta+2\Delta,0),$$

where $t = n\Delta$.

For fixed $x, z \in \mathbf{R}$, $t > 0$, we want to estimate

$$U^\Delta(x,t)-U^\Delta(z,t) = -2\Delta\log\left(\frac{\sum_{l=0}^{n}\binom{n}{l}e^{-\frac{1}{\Delta}U^\Delta(x-n\Delta+2l\Delta,0)}}{\sum_{l=0}^{n}\binom{n}{l}e^{-\frac{1}{\Delta}U^\Delta(z-n\Delta+2l\Delta,0)}}\right), \tag{3.2.4}$$

where we've used $U^{\Delta}(\cdot,0) = -2\Delta\log(V^{\Delta}(\cdot,0))$. For the sake of a simpler formula, we assume that (x,t) and (z,t) are always on the grid points as the mesh refines.

The following counterpart of Lemma 2.1 is crucial in establishing the ordering of $u^{\Delta}(\cdot,t_c)$ and $u(\cdot,t_c)$; therefore, we can estimate the L^1 difference of the two using (3.2.4).

LEMMA 3.4. *Let u_0 be a smooth and bounded function satisfying*

(B.1) $f'_L(u_0(\xi)) = -\frac{\xi}{t_c} + a\xi^3 + b\xi^4 + O(\xi^5)$ *for* $|\xi| < \delta$ *where* $a > 0$.

(B.2) $\xi = 0$ *is the point corresponding to the first spontaneous formation of shocks; that is,* $\frac{d}{d\xi} f'_L(u_0(\xi)) > -\frac{1}{t_c}$ *for all* $\xi \neq 0$.

(B.3) $f'_L(u_0)$ *is antisymmetric, and thus so is* u_0*:* $u_0(-\xi) = -u_0(\xi)$.

(B.4) $f'_L(u_0(\cdot))$*, and, therefore,* $u_0(\cdot)$ *is monotonely nonincreasing.*

(B.5) $f'_L(u_0(\cdot))$ *is concave on* $\xi < 0$*, and, therefore, by Assumption* (B.3)*, convex on* $\xi > 0$.

Then

$$u(x,\Delta) \geq u^{\Delta}(x,\Delta) \quad x < 0,\ (x,\Delta) \text{ on the grids.}$$

By induction and the monotonicity of Lax–Friedrichs scheme, $u(x,t) \geq u^{\Delta}(x,t)$ *for all* $x < 0$, $t > 0$, (x,t) *on the grids.*

Proof. Let $A = (x,\Delta), B = (x-\Delta,0), C = (x+\Delta,0)$, and $D = (x,0)$ be four-grid points on $x < 0$; then

- $u^{\Delta}_A = g(u_B,u_C) \equiv \frac{1}{2}(u_B + u_C) - \frac{1}{2}(f_L(u_C) - f_L(u_B))$;
- $u_B \geq u_C$, and $u_A = u_E$ for some (unique) point E on the line segment $\bar{BD}$.

Denote by $m = f'_L(u_E) = distance(D,E)/\Delta$, and define, for $0 \leq \theta \leq 1$, a family of functions $v(\theta,\xi)$ on $\bar{BC}$ by

$$f'_L(v(\theta,\xi)) = m + \theta(f'_L(u_0(\xi)) - m), \quad 0 \leq \theta \leq 1,\ x-\Delta \leq \xi \leq x+\Delta.$$

Obviously, $v(0,\xi) = u_E$ and $v(1,\xi) = u_0(\xi)$.

Now let $h(\theta) = g(v(\theta,x-\Delta), v(\theta,x+\Delta))$; then $h(0) = u_E$ and $h(1) = g(u_B,u_C) = u^{\Delta}_A$. A direct computation gives

$$\frac{dh}{d\theta} = \frac{1}{2}(\alpha + \theta\alpha\beta + m\beta),$$

where $\alpha = f'(u_B)+f'(u_C)-2m < 0$ and $\beta = f'(u_B)-f'(u_C) > 0$. Due to concavity of $f'(u_0(\cdot))$, the graph of $f'(u_0(\cdot))$ lies above the line joining $(B,f'(u_B))$ and $(C,f'(u_C))$; therefore, $\alpha + m\beta \leq 0$. Thus $\frac{dh}{d\theta} \leq 0$ and $u_A \geq u^{\Delta}_A$. $\square$

Now we come back to estimate the leading order term of (3.2.4) using Stirling's formula

$$n! = \left(\frac{n-1}{e}\right)^{n-1} (2\pi(n-1))^{\frac{1}{2}} + \cdots. \tag{3.2.5}$$

After elementary calculations, we have

$$U^{\Delta}(x,t) - U^{\Delta}(z,t) = -2\Delta\log\left(\frac{\displaystyle\sum_{l=0}^{n} e^{-\frac{1}{\Delta}[(tF(\frac{x-y}{t})+U_0(y,o))+E_1]}}{\displaystyle\sum_{l=0}^{n} e^{-\frac{1}{\Delta}[(tF(\frac{z-y}{t})+U_0(y,o))+E_2]}}\right), \tag{3.2.6}$$

where $F(s) = \log(1-s^2) + s\log(\frac{1+s}{1-s})$, $t = n\Delta$, and $x - y = t - 2l\Delta$. E_1 and E_2 are the errors introduced by Stirling's formula, and are of lower order.

We next replace sums by integrals. Again, the errors are of lower order since the integrals are at least $O(\Delta^{\frac{1}{2}})$ as we saw in section 3.1. This leads to

$$U^\Delta(x,t) - U^\Delta(z,t) = -2\Delta \log\left(\frac{\displaystyle\int_{x-t}^{x+t} e^{-\frac{1}{2\Delta}[(tF(\frac{x-y}{t})+\int^y u_0(y')dy')+E_1]}dy}{\displaystyle\int_{z-t}^{z+t} e^{-\frac{1}{2\Delta}[(tF(\frac{z-y}{t})+\int^y u_0(y')dy')+E_2]}dy}\right) + \cdots.$$
(3.2.7)

At $x = 0$, $t = t_c$, the integrand of the numerator has a quartic phase at its maximum $y = 0$, while the integrand of the denominator has a quadratic phase at $z =$, say, -1, $t = t_c$. Therefore,

$$U^\Delta(0,t_c) - U^\Delta(-1,t_c) = \int_{-1}^{0} u(.,t_c) - 2\Delta\log\left(\frac{\Delta^{\frac{1}{4}}}{\Delta^{\frac{1}{2}}}\right) + \cdots,$$

and

$$\|u(\cdot,t_c) - u^\Delta(\cdot,t_c)\|_{L^1(\Delta x;[-1,0])} \sim \frac{1}{2}\Delta|\log\Delta|.$$

The generalization to nonantisymmetric initial data is the same as for Burgers's equation in the previous subsection.

Remark 2.

1. The case $t \neq t_c$ can be proved in the same way; see [2] for a discrete version of Lemma 2.2. The discrete analogue of the comparison lemma is an immediate consequence of monotonicity.

2. Although we only carry out the analysis for the most dissipative first-order scheme, namely, the Lax–Friedrichs scheme, the same argument shows that even the upwind scheme cannot do better. Since the same transform applies to the upwind scheme with the flux function $f(u) = -\log(a+be^{-u})$, $a, b > 0, a+b = 1$. Even though we don't have symmetry in this case, we still have the lower bound for free:

$$\|u(\cdot,t_c) - u^\Delta(\cdot,t_c)\|_{L^1(\Delta x;[x_c-1,x_c])} \geq |U(x_c,t_c) - U^\Delta(x_c-1,t_c)| = O(\Delta|\log\Delta|)$$
(3.2.8)

3. Since $f'(u) = \frac{be^{-u}}{a+be^{-u}} > 0$, the Godunov scheme reduces to upwind scheme. Therefore, (3.2.8) also holds for Godunov scheme with the same family of flux.

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