

SPECTRAL VISCOSITY METHOD WITH GENERALIZED HERMITE FUNCTIONS AND ITS SPECTRAL FILTER FOR NONLINEAR CONSERVATION LAWS

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Abstract. In this paper, we propose a new spectral viscosity method for the solution of nonlinear scalar conservation laws in the whole line. The proposed method is based on the generalized Hermite functions. It is shown rigorously that this scheme converges to the unique entropy solution by using compensated compactness arguments. The numerical experiments of the inviscid Burger’s equation support our result. Moreover, we step further to discuss the postprocessing method of the approximate solution obtained by our method to eliminate the oscillation away from the discontinuity. We suggest a new spectral filter defined by a kernel function in the physical space, and show the spectral accuracy at the points away from the discontinuity.

Key words. spectral viscosity method, generalized Hermite functions, nonlinear conservation laws, compensated compactness arguments, spectral filter

AMS subject classifications. 65M70, 35L65, 65M10

1. Introduction. The spectral methods [9] approximate the exact solution of partial differential equations by seeking an “good” projection in the linear subspace spanned by various orthogonal systems of special functions. The resulting spectral accuracy is highly preferred than any other numerical method, especially when the solution is known to be globally smooth enough. Therefore, they are very appropriate in the case of elliptic and parabolic equations, thanks to the regularization properties of the operators. However, it is well known that the solution to the nonlinear conservation laws may develop spontaneous jump discontinuity, i.e., shock waves. This irregularity of the solution destroys not only the accuracy of the spectral approximations at the point of discontinuity, but also that in the entire computational domain. It causes the oscillations throughout the domain, which is the so-called Gibb’s phenomenon. Moreover, the instability is induced in the nonlinear case. It is shown in [24] that the usual spectral approximate solution may not converge to the entropy solution, the physically relevant one.

Despite all these deficiencies, many mathematicians pay their effort to deal with these problems. The problems caused by the irregularity has already been solved for piecewise smooth functions in bounded domain or periodic piecewise smooth function in unbounded domain by filter techniques or reconstruction methods such as the Gegenbauer partial sum, see details in a series of papers [12], [11], [10], [27] and references therein. And the instability of the usual spectral approximations can be avoided by introducing the vanishing viscosity, which was first established by E. Tadmor [23]. The main ingredient of the spectral viscosity method is the use of high-frequencies diffusion which stabilizes the spectral computation without sacrificing spectral accuracy. The periodic spectral method has been further investigated in [17], [24] and [19], etc. The nonperiodic Legendre spectral viscosity method is first introduced by Y. Maday, et. al. [16]. And H. Ma proposed the nonperiodic Chebyshev-Legendre spectral viscosity method in [14], [15]. For more literatures related to the spectral viscosity methods with various orthogonal basis in bounded domain, we refer the readers to [6], [8], [13] and references therein.

In the real world, as we know, a large number of physical problems are modelled in unbounded domain. During the past two decades, more and more attentions were paid to numerical solutions of differential equations defined in unbounded domains. Among the existing literature, the Hermite and Laguerre spectral methods are the most commonly used approaches based on orthogonal polynomials in infinite interval, referring to [7], [28]. Although the Hermite polynomials appear to be a natural choice of orthogonal basis of $L^2(\mathbb{R})$, it is not as popular as Fourier series and Chebyshev polynomials, due to its poor resolution (see [9]) and the lack of the analogue of fast Fourier transformation (FFT), see [4]. However, it is shown in [2] that the poor resolution can be remedied by a suitable choice

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of scaling factor. Some further investigations on the scaling factor can be found in [25] and also in Chapter 7, [21]. Recently, a guideline of choosing the suitable scaling factors for Gaussian/super-Gaussian functions is summarized by the author and S. S.-T. Yau in [18], in which they use the Hermite spectral method to resolve the conditional density function of the states in nonlinear filtering problems.

The literatures on the spectral method in unbounded domains have already been not as rich as those in bounded domains, let alone the spectral viscosity method in unbounded domains. As far as we know, J. Aguirre and J. Rivas [1] is the only paper that considered the spectral viscosity method based on the Hermite functions. However, they defined the Hermite functions in the weighted $L_w^2(\mathbb{R})$, where $w(x) = e^{-x^2}$. And no scaling factor is introduced. This essentially causes their involved theoretical proof of the convergence rate of their proposed scheme under more restrictive conditions. It is even worse when they try to implement the spectral one numerically (the numerical experiments are done in the pseudospectral case in their paper), since the Fourier-Hermite coefficients can't be resolved accurately enough.

In this paper, let us revisit the nonlinear scalar conservation laws in one dimension posed in the whole line

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $f \in C^1$ is a smooth nonlinear function and $u_0 \in L^\infty(\mathbb{R})$. In general, the spontaneous jump discontinuity may be developed. Therefore, we can't expect the classical solutions to this problem. Moreover, we restrict ourselves to the physically relevant weak solution, the entropy solution, by imposing the entropy condition

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0 \quad (1.2)$$

in the sense of distributions, for all entropy pairs (U, F) , with $U \in C^2$ convex and $F'(u) = U'(u)f'(u)$, see [20].

We propose a new Hermite spectral viscosity method, which yields an approximate solution converging to the unique entropy solution. The method in this paper is superior to the one proposed in [1] in the following aspects:

1. One restrictive condition ((4.13) in [1]) can be replaced by the boundedness of the approximate solution over time and space in the proof of the convergence rate of our scheme.
2. The viscosity term introduced in this paper stabilized our scheme, due to its symmetry and positivity; while the stability of the scheme in [1] is not guaranteed.
3. The spectral viscosity method can be implemented easily, with the help of the scaling factor. The better resolution and fewer oscillations are retained, with much smaller truncation terms N , compared with that used in [1], even without viscosity (Figure 5.1).

Besides the new Hermite spectral viscosity methods, we also introduced a spectral filter defined by the kernel function in physical space. It is based on the idea of localizing the information in obtaining the Fourier-Hermite coefficients. The numerical experiments support our expectation that it improves the resolution greatly at the points away from the discontinuity.

The paper is organized as follows. In section 2, we give the definition of the generalized Hermite functions and their properties. We propose the new Hermite spectral viscosity method in section 3. The convergence rate of our scheme has been rigorously shown under some reasonable conditions by using the compensated compactness arguments. Section 4 is devoted to develop the spectral filter which can post-process the oscillations caused by the Gibb's phenomenon away from the point of discontinuity. Finally, we present some numerical experiments of the inviscid Burger's equation in section 5. Our theoretical results are verified by the experiments.

2. Generalized Hermite functions. In this section, we introduce the generalized Hermite functions and derive some properties which are inherited from the Hermite polynomials. To establish

the convergence rate of the Hermite spectral method, we shall also state the convergence rate of the orthogonal approximation.

Let $L^2(\mathbb{R})$ be the Lebesgue space, equipped with the norm $\|\cdot\| = (\int_{\mathbb{R}} |\cdot|^2 dx)^{\frac{1}{2}}$ and the scalar product $\langle \cdot, \cdot \rangle$. In the sequel, we shall follow the convection in the asymptotic analysis, $a \sim b$ means that there exists some generic constants $C_1, C_2 > 0$ such that $C_1 a \leq b \leq C_2 a$; $a \lesssim b$ means that there exists some generic constant $C_3 > 0$ such that $a \leq C_3 b$.

Let $\mathcal{H}_n(x)$ be the physical Hermite polynomials, i.e., $\mathcal{H}_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$, $n \geq 0$. The three-term recurrence

$$\mathcal{H}_0 \equiv 1, \quad \mathcal{H}_1(x) = 2x \quad \text{and} \quad \mathcal{H}_{n+1}(x) = 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x). \quad (2.1)$$

is more handy in implementation. One of the well-known and useful fact of Hermite polynomials is that they are mutually orthogonal with respect to the weight $w(x) = e^{-x^2}$. We define the generalized Hermite functions with the scaling factor $\alpha > 0$ as

$$H_n^\alpha(x) = \left(\frac{\alpha}{2^n n! \sqrt{\pi}} \right)^{\frac{1}{2}} \mathcal{H}_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}, \quad (2.2)$$

for $n \geq 0$. It is readily to derive the following properties for the generalized Hermite functions (2.2):

- The value of $H_n^\alpha(x)$ at 0 is

$$H_n^\alpha(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}} \frac{\alpha^{\frac{1}{2}} (n-1)!!}{(n!)^{\frac{1}{2}} \pi^{\frac{1}{4}}}, & \text{if } n \text{ is even.} \end{cases}$$

Moreover, with the Stirling approximation, we have

$$H_{2k}^\alpha(0) \sim (-1)^k \alpha^{\frac{1}{2}} k^{-\frac{1}{4}}, \quad (2.3)$$

for $k \gg 1$.

- The $\{H_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis of $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} H_n^\alpha(x) H_m^\alpha(x) dx = \delta_{nm}, \quad (2.4)$$

where δ_{nm} is the Kronecker function.

- $H_n^\alpha(x)$ is the n^{th} eigenfunction of the following Sturm-Liouville problem

$$e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} u(x))) + \lambda_n u(x) = 0, \quad (2.5)$$

with the corresponding eigenvalue

$$\lambda_n = 2\alpha^2 n. \quad (2.6)$$

For conciseness, let us denote the Sturm-Liouville operator as

$$\mathcal{L}_\alpha(\circ) = -e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} \circ)). \quad (2.7)$$

- By convention, $H_n^\alpha \equiv 0$, for $n < 0$. For $n \geq 0$, the three-term recurrence is inherited from the Hermite polynomials:

$$x H_n^\alpha(x) = \frac{\sqrt{\lambda_{n+1}}}{2\alpha^2} H_{n+1}^\alpha(x) + \frac{\sqrt{\lambda_n}}{2\alpha^2} H_{n-1}^\alpha(x). \quad (2.8)$$

- The derivative of $H_n^\alpha(x)$ with respect to x

$$\partial_x H_n^\alpha(x) = -\frac{\sqrt{\lambda_{n+1}}}{2} H_{n+1}^\alpha(x) + \frac{\sqrt{\lambda_n}}{2} H_{n-1}^\alpha(x). \quad (2.9)$$

For convenience, let $\mathcal{D}_x = \partial_x + \alpha^2 x$. Then

$$\mathcal{D}_x H_n^\alpha(x) = \sqrt{2\alpha^2 n} H_{n-1}^\alpha(x) = \sqrt{\lambda_n} H_{n-1}^\alpha(x). \quad (2.10)$$

■ The “orthogonality” of $\{xH_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$, $\{\partial_x H_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$ and $\{\mathcal{D}_x H_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$ are list as following:

$$\int_{\mathbb{R}} x^2 H_n^\alpha(x) H_m^\alpha(x) dx = \begin{cases} \frac{\lambda_{n+1} + \lambda_n}{4\alpha^4}, & \text{if } m = n \\ \frac{\sqrt{\lambda_{l-1}\lambda_l}}{4\alpha^4}, & l = \max\{m, n\}, \text{ if } |n - m| = 2 \\ 0, & \text{otherwise} \end{cases} \quad (2.11)$$

$$\int_{\mathbb{R}} \partial_x H_n^\alpha(x) \partial_x H_m^\alpha(x) dx = \begin{cases} \frac{\lambda_{n+1} + \lambda_n}{4}, & \text{if } m = n \\ -\frac{\sqrt{\lambda_l \lambda_{l-1}}}{4}, & l = \max\{n, m\}, \text{ if } |n - m| = 2 \\ 0, & \text{otherwise} \end{cases} \quad (2.12)$$

and

$$\int_{\mathbb{R}} \mathcal{D}_x H_n^\alpha(x) \mathcal{D}_x H_m^\alpha(x) dx = 2\alpha^2 n \delta_{nm} = \lambda_n \delta_{nm}, \quad (2.13)$$

where δ_{nm} is the Kronecker function.

Any function $u(x) \in L^2(\mathbb{R})$ can be written in the form

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n H_n^\alpha(x), \quad (2.14)$$

with

$$\hat{u}_n = \int_{\mathbb{R}} u(x) H_n^\alpha(x) dx. \quad (2.15)$$

where $\{\hat{u}_n\}_{n=0}^{\infty}$ are the Fourier-Hermite coefficients.

Let us denote the linear subspace of $L^2(\mathbb{R})$ spanned by the first N Hermite functions by

$$\mathcal{R}_N := \text{span}\{H_0^\alpha(x), \dots, H_N^\alpha(x)\}. \quad (2.16)$$

REMARK 2.1. *The norms $\|\partial_x \phi\|$ and $\|\mathcal{D}_x \phi\|$ are equivalent, for $\phi \in \mathcal{R}_N$. Let us consider*

$$\begin{aligned} \|\partial_x \phi\|^2 &= \left\| \sum_{k=0}^N \hat{\phi}_k \left(-\frac{\sqrt{\lambda_{k+1}}}{2} H_{k+1}^\alpha + \frac{\sqrt{\lambda_k}}{2} H_{k-1}^\alpha \right) \right\|^2 \\ &= \frac{1}{4} \sum_{k=0}^N \left\{ \hat{\phi}_k^2 \lambda_{k+1} - \hat{\phi}_{k+2} \hat{\phi}_k \sqrt{\lambda_{k+2} \lambda_{k+1}} - \hat{\phi}_k \hat{\phi}_{k-2} \sqrt{\lambda_k \lambda_{k-1}} + \hat{\phi}_k^2 \lambda_k \right\}, \end{aligned}$$

where $\hat{\phi}_k = 0$, for all $k > N$ or $k < 0$. Recall that

$$\left| \hat{\phi}_{k+2} \hat{\phi}_k \sqrt{\lambda_{k+2} \lambda_{k+1}} \right| \leq \frac{1}{2} (\hat{\phi}_{k+2}^2 \lambda_{k+2} + \hat{\phi}_k^2 \lambda_{k+1}).$$

We arrive at

$$\|\partial_x \phi\|^2 \sim \sum_{k=0}^N \left\{ \hat{\phi}_k^2 (\lambda_{k+1} + \lambda_k) \right\} \sim \sum_{k=0}^N \hat{\phi}_k^2 \lambda_k = \|\mathcal{D}_x \phi\|^2. \quad (2.17)$$

Furthermore, $\|\partial_x \phi\|^2$ and $\langle x\phi, \partial_x \phi \rangle$ are also equivalent, for all $\phi \in \mathcal{R}_N$, due to the similar argument.

We define the L^2 -orthogonal projection $P_N^\alpha : L^2(\mathbb{R}) \rightarrow \mathcal{R}_N$, given $v \in L^2(\mathbb{R})$,

$$\langle v - P_N^\alpha v, \phi \rangle = 0,$$

for all $\phi \in \mathcal{R}_N$. More precisely,

$$P_N v(x) = \sum_{n=0}^N \hat{v}_n H_n^\alpha(x),$$

where \hat{v}_n , $n = 0, \dots, N$, are the Fourier-Hermite coefficients defined in (2.14).

The error estimate of the orthogonal projection onto \mathcal{R}_N is readily shown in Theorem 4.2, [22] for $\alpha = 1$ and it can be trivially extended for $\alpha > 0$ that

LEMMA 2.1. *For any $\mathcal{D}_x^m u \in L^2(\mathbb{R})$ with $m \geq 0$,*

$$\|\mathcal{D}_x^l(u - P_N^\alpha u)\| \lesssim \alpha^{l-m} N^{\frac{l-m}{2}} \|\mathcal{D}_x^m u\|, \quad 0 \leq l \leq m.$$

In the sequel, the superscript α in P_N^α will be dropped if no confusion will arise.

3. The Hermite spectral viscosity method. It is well known that the entropy solution to (1.1) can be obtained as the limit when the artificial introduced viscosity term vanishes. Based on this fact, we introduce a more appropriate viscosity term $\epsilon \mathcal{L}_\alpha$ (defined in (2.7)) to replace the Laplacian ϵu_{xx} .

One can show the following result for the entropy solution to (1.1) by using the properties of the operator \mathcal{L}_α and the compensated compactness arguments.

THEOREM 3.1. *Let $f \in C^1$ be a nonlinear function such that $f'(0) = 0$, $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ and u_ϵ the solution of*

$$\begin{cases} \partial_t u_\epsilon + \partial_x f(u_\epsilon) + \epsilon \mathcal{L}_\alpha u_\epsilon = 0, & x \in \mathbb{R}, t > 0, \\ u_\epsilon(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (3.1)$$

Then, for any $p \geq 1$ and any $\Omega \subset \mathbb{R} \times (0, \infty)$ open and bounded, u_ϵ converges in $L^p(\Omega)$ to the unique entropy solution to (1.1) when ϵ tends to 0.

By taking this into account, we define the spectral viscosity method to the unique entropy solution of (1.1) as $u_N(x, t) = \sum_{k=0}^N \hat{u}_k(t) H_k^\alpha(x)$ such that

$$\begin{cases} \langle \partial_t u_N + \partial_x (P_{N+1} f(u_N)), \varphi \rangle + \epsilon_N \langle \mathcal{L}_\alpha (Q_{m_N} u_N), Q_{m_N} \varphi \rangle = 0, & x \in \mathbb{R}, t \in (0, T), \\ u_N(x, 0) = P_N u_0(x), & x \in \mathbb{R}, \end{cases} \quad (3.2)$$

for any $\varphi(x) \in \mathcal{R}_N$ (defined in (2.16)), where \mathcal{L}_α is the Sturm-Liouville operator defined in (2.7), ϵ_N is a positive parameter depending on N tending to 0 as N tends to ∞ , and Q_{m_N} is a viscosity operator which modifies only the high modes of the Fourier-Hermite expansion, that is,

$$Q_{m_N} \left(\sum_{k=0}^N \hat{\phi}_k(t) H_k^\alpha(x) \right) = \sum_{k=0}^N \hat{q}_k \hat{\phi}_k(t) H_k^\alpha(x), \quad (3.3)$$

where

$$\begin{cases} \hat{q}_k = 0, & \text{if } k \leq m_N \\ 1 - \frac{m_N}{k} \leq \hat{q}_k \leq 1, & \text{if } k > m_N \end{cases}, \quad (3.4)$$

and $m_N < N$ is a positive integer that tends to ∞ with N .

LEMMA 3.2. *Let Q_{m_N} be defined as in (3.3), (3.4). Then*

$$\|\mathcal{D}_x \phi\|^2 \lesssim \|\mathcal{D}_x Q_{m_N} \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2, \quad (3.5)$$

and

$$\|\mathcal{D}_x Q_{m_N} \phi\|^2 \lesssim \|\mathcal{D}_x \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2. \quad (3.6)$$

for all $\phi \in \mathcal{R}_N$.

Proof. Let us start to show (3.5). And (3.6) follows from the similar argument. Let $\phi = \sum_{k=0}^N \hat{\phi}_k H_k^\alpha(x)$, and $R_{m_N} = I - Q_{m_N}$, where I is the identity operator. Then

$$\|\mathcal{D}_x \phi\|^2 \lesssim \|\mathcal{D}_x Q_{m_N} \phi\|^2 + \|\mathcal{D}_x R_{m_N} \phi\|^2. \quad (3.7)$$

We split ϕ in dyadic parts $\phi(x) = \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha(x) + \sum_{j=1}^J \phi^j(x)$, where

$$\phi^j(x) = \sum_{k=2^{j-1}m_N+1}^{2^j m_N} \hat{\phi}_k H_k^\alpha(x),$$

$j = 1, \dots, J$. Here $J = \log_2 \left(\frac{N}{m_N} \right) + 1$ and $\hat{\phi}_k = 0$ for $k = N+1, \dots, 2^J m_N$. From the orthogonality relation (2.13), one has

$$\|\mathcal{D}_x R_{m_N} \phi\|^2 = \left\| \mathcal{D}_x R_{m_N} \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2 + \sum_{j=1}^J \|\mathcal{D}_x R_{m_N} \phi^j\|^2. \quad (3.8)$$

We bound each summand above by using the fact that given R a linear operator defined in \mathcal{R}_N such that

$$R \left(\sum_{k=0}^N \hat{\phi}_k H_k^\alpha(x) \right) = \sum_{k=0}^N \hat{r}_k \hat{\phi}_k H_k^\alpha(x),$$

where $\hat{r}_0, \dots, \hat{r}_N$ are real numbers. Then for all $\phi \in \mathcal{R}_N$,

$$\|\mathcal{D}_x R \phi\|^2 \stackrel{(2.13)}{=} \sum_{k=0}^N \hat{r}_k^2 \hat{\phi}_k^2 \lambda_k \leq \left(\sum_{k=0}^N \hat{r}_k^2 \lambda_k \right) \left(\sum_{k=0}^N \hat{\phi}_k^2 \right) = \left(\sum_{k=0}^N \hat{r}_k^2 \lambda_k \right) \|\phi\|^2. \quad (3.9)$$

Since $\hat{q}_k = 0$, for $k \leq m_N$,

$$\left\| \mathcal{D}_x R_{m_N} \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2 \stackrel{(3.9)}{\leq} \left(\sum_{k=0}^{m_N} (1 - \hat{q}_k)^2 \lambda_k \right) \left(\sum_{k=0}^{m_N} \hat{\phi}_k^2 \right) \lesssim \alpha^2 m_N^2 \left(\sum_{k=0}^{m_N} \hat{\phi}_k^2 \right) = \alpha^2 m_N^2 \left\| \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2. \quad (3.10)$$

For the second summand on the right-hand side of (3.8), since $\hat{q}_k \geq 1 - \frac{m_N}{k}$, we have, for any $j = 1, \dots, J$,

$$\|\mathcal{D}_x R_{m_N} \phi^j\|^2 \stackrel{(3.9)}{\lesssim} \left(\sum_{k=2^{j-1}m_N+1}^{2^j m_N} (1 - \hat{q}_k)^2 \lambda_k \right) \|\phi^j\|^2 \lesssim \alpha^2 m_N^2 \sum_{k=2^{j-1}m_N+1}^{2^j m_N} \frac{1}{k} \|\phi^j\|^2 \lesssim \alpha^2 m_N^2 \|\phi^j\|^2. \quad (3.11)$$

Combine (3.10) and (3.11), we obtain that

$$\|\mathcal{D}_x R_{m_N} \phi\|^2 \lesssim \alpha^2 m_N^2 \left(\left\| \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2 + \sum_{j=1}^J \|\phi^j\|^2 \right) \lesssim \alpha^2 m_N^2 \|\phi\|^2. \quad (3.12)$$

Substituting the above equation back to (3.7), we get the desired result (3.5). And (3.6) can be obtained similarly, with the fact that

$$\|\mathcal{D}_x Q_{m_N} \phi\|^2 \lesssim \|\mathcal{D}_x \phi\|^2 + \|\mathcal{D}_x R_{m_N} \phi\|^2 \stackrel{(3.12)}{\lesssim} \|\mathcal{D}_x \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2.$$

□

3.1. A priori estimates. Let us start with some a priori estimates on the approximate solution u_N .

LEMMA 3.3. *Let $f \in C^2(\mathbb{R})$ be such that $f'(0) = 0$, $u_0 \in L^2(\mathbb{R})$, $T > 0$, Q_{m_N} is given in (3.3), (3.4), and $u_N : [0, T] \rightarrow \mathcal{R}_N$ the solution of (3.2). Let us assume that there is a positive constant C , independent of N and α , such that*

$$\|u_N\|_\infty \leq C, \quad (3.13)$$

and $\epsilon_N m_N^2 = o(1)$. Then

$$\|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0, T; L^2(\mathbb{R}))} \lesssim \frac{1}{\sqrt{\epsilon_N}}, \quad (3.14)$$

and

$$\|\partial_t u_N\|_{L^2(0, T; L^2(\mathbb{R}))} + \|\partial_x u_N\|_{L^2(0, T; L^2(\mathbb{R}))} \lesssim \frac{1}{\sqrt{\epsilon_N}}, \quad (3.15)$$

where the norms are defined as $\|\circ\|_\infty = \|\circ\|_{L^\infty(\mathbb{R} \times (0, T))} = \sup_{\mathbb{P}_{\mathbb{R} \times [0, T]}} |\circ|$ and $\|\circ\|_{L^2(0, T; L^2(\mathbb{R}))} := \int_0^T \|\circ\|^2 dt$.

Proof. Let us choose $\varphi = u_N \in \mathcal{R}_N$ in (3.2) and it yields that

$$\begin{aligned} 0 &\stackrel{(2.7)}{=} \int_{\mathbb{R}} u_N \partial_t u_N dx + \int_{\mathbb{R}} \partial_x (P_{N+1} f(u_N)) u_N dx \\ &\quad - \epsilon_N \int_{\mathbb{R}} e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} Q_{m_N} u_N)) Q_{m_N} u_N dx \\ &= \frac{1}{2} \frac{d}{dt} \|u_N\|^2 + \int_{\mathbb{R}} \partial_x (P_{N+1} f(u_N)) u_N dx + \epsilon_N \|\mathcal{D}_x Q_{m_N} u_N\|^2. \end{aligned} \quad (3.16)$$

The last term on the right-hand side of (3.16) is followed by integration by parts, i.e.,

$$\begin{aligned} \langle \mathcal{L}_\alpha f, g \rangle &= - \int_{\mathbb{R}} e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} f(x))) g(x) dx \\ &= \int_{\mathbb{R}} e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} f(x)) \partial_x (e^{\frac{1}{2}\alpha^2 x^2} g(x)) dx = \int_{\mathbb{R}} \mathcal{D}_x f(x) \mathcal{D}_x g(x) dx, \end{aligned} \quad (3.17)$$

since

$$\partial_x (e^{\frac{1}{2}\alpha^2 x^2} f(x)) = e^{\frac{1}{2}\alpha^2 x^2} (\partial_x + \alpha^2 x) f(x) = e^{\frac{1}{2}\alpha^2 x^2} \mathcal{D}_x f(x).$$

The second term on the right-hand side of (3.16) gives

$$\begin{aligned} \int_{\mathbb{R}} \partial_x (P_{N+1} f(u_N)) u_N dx &= \int_{\mathbb{R}} P_{N+1} \partial_x (f(u_N)) u_N dx \\ &\quad - \frac{1}{2} \sqrt{\lambda_{N+2}} \int_{\mathbb{R}} \left[\widehat{f(u_N)}_{N+1}(t) H_{N+2}^\alpha(x) + \widehat{f(u_N)}_{N+2}(t) H_{N+1}^\alpha(x) \right] u_N dx \\ &= \int_{\mathbb{R}} P_{N+1} \partial_x (f(u_N)) u_N dx = \int_{\mathbb{R}} \partial_x (f(u_N)) u_N dx, \end{aligned} \quad (3.18)$$

where the first equality follows from the fact that

$$P_N \partial_x \phi(x, t) - \partial_x P_N \phi(x, t) = \frac{1}{2} \sqrt{\lambda_{N+1}} \left[\hat{\phi}_N(t) H_{N+1}^\alpha(x) + \hat{\phi}_{N+1}(t) H_N^\alpha(x) \right]. \quad (3.19)$$

and the last equality holds due to the orthogonality of generalized Hermite function. Let F be a primitive of $u f'(u)$. Integrating (3.16) from 0 to T and taking the assumption $f'(0) = 0$ into account, we obtain that

$$\|u_N\|^2(T) + 2\epsilon_N \int_0^T \|\mathcal{D}_x Q_{m_N} u_N\|^2 dt = \|P_N u_0\|^2 \leq \|u_0\|^2.$$

Hence, (3.14) is followed immediately.

Next, we set $\varphi = \partial_t u_N$ in the spectral approximation (3.2).

$$\|\partial_t u_N\|^2 + \langle \partial_x P_{N+1} f(u_N), \partial_t u_N \rangle + \frac{1}{2} \epsilon_N \frac{d}{dt} \|\mathcal{D}_x(Q_{m_N} u_N)\|^2 = 0, \quad (3.20)$$

where the last term on the left-hand side of (3.20) is followed by

$$\begin{aligned} \langle \mathcal{L}_\alpha g, \partial_t g \rangle &= - \int_{\mathbb{R}} e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} g)) \cdot \partial_t g dx \\ &= \int_{\mathbb{R}} e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} g) \partial_t \partial_x (e^{\frac{1}{2}\alpha^2 x^2} g) dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\partial_x (e^{\frac{1}{2}\alpha^2 x^2} g)|^2 e^{-\alpha^2 x^2} dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |(\partial_x + \alpha^2 x)g|^2 dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\mathcal{D}_x g|^2 dx = \frac{1}{2} \frac{d}{dt} \|\mathcal{D}_x g\|^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\partial_t u_N\|^2 + \frac{1}{2} \epsilon_N \frac{d}{dt} \|\mathcal{D}_x(Q_{m_N} u_N)\|^2 &\stackrel{(3.19)}{=} - \langle P_{N+1} \partial_x f(u_N), \partial_t u_N \rangle \\ &\quad + \frac{1}{2} \sqrt{\lambda_{N+2}} \langle \hat{\phi}_{N+1} H_{N+2}^\alpha(x) + \hat{\phi}_{N+2} H_{N+1}^\alpha(x), \partial_t u_N \rangle \\ &= - \langle P_{N+1} \partial_x f(u_N), \partial_t u_N \rangle \\ &\leq \|P_{N+1} \partial_x f(u_N)\| \cdot \|\partial_t u_N\| \leq C \|\partial_x u_N\|^2 + \frac{1}{2} \|\partial_t u_N\|^2, \end{aligned} \quad (3.21)$$

where the second equality on the right-hand side of (3.21) is due to the fact that $\partial_t u_N \in \mathcal{R}_N$ and the orthogonality of the generalized Hermite functions.

Integrating (3.20) from 0 to T , we have

$$\|\partial_t u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \leq \epsilon_N \|\mathcal{D}_x(Q_{m_N} u_N)\|^2(0) - \epsilon_N \|\mathcal{D}_x(Q_{m_N} u_N)\|^2(T) + C \|\partial_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2. \quad (3.22)$$

From Remark 2.1 and Lemma 3.2, we have

$$\begin{aligned} \|\partial_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 &\lesssim \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 + \alpha^2 m_N^2 \|u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \\ &\stackrel{(3.14)}{\lesssim} \frac{1}{\epsilon_N} + \alpha^2 m_N^2 \lesssim \frac{1}{\epsilon_N}, \end{aligned} \quad (3.23)$$

if $\alpha^2 m_N^2 \epsilon_N = o(1)$. Substituting (3.23) back to (3.22), (3.15) follows immediately. \square

3.2. The convergence of the spectral viscosity approximation. Let $\Omega = I \times [0, T]$ be an open and bounded subset of $\mathbb{R} \times [0, T]$ and define

$$\langle g, h \rangle_\Omega := \int_0^T \langle g, h \rangle_{L^2(I)} dt; \quad \|g\|_\Omega^2 := \int_0^T \|g\|_{L^2(I)}^2 dt,$$

where I is some interval in \mathbb{R} .

Before we proceed to prove the convergence theorem, we need the following lemma:

LEMMA 3.4. *Let $\varphi \in H_0^1(\Omega)$ and $\varphi_N = P_N \varphi \in \mathcal{R}_N$, then*

$$\|\partial_x \varphi_N\|_\Omega + \alpha \sqrt{N} \|\varphi - \varphi_N\|_\Omega \lesssim \|\partial_x \varphi\|_\Omega. \quad (3.24)$$

Proof. From Lemma 2.1, we obtain that

$$\alpha \sqrt{N} \|\varphi - \varphi_N\| \lesssim \|\mathcal{D}_x \varphi\| \sim \|\partial_x \varphi\|.$$

Integrating from 0 to T , it yields that

$$\alpha\sqrt{N}\|\varphi - \varphi_N\|_{\Omega} \leq \alpha\sqrt{N}\|\varphi - \varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim \|\partial_x\varphi\|_{L^2(0,T;L^2(\mathbb{R}))} = \|\partial_x\varphi\|_{\Omega},$$

since $\varphi \in H_0^1(\Omega)$. And from Remark 2.1 and the fact that

$$\mathcal{D}_x P_N \varphi = P_{N-1} \mathcal{D}_x \varphi, \quad (3.25)$$

for all $\varphi \in L^2(\mathbb{R})$, we have

$$\|\partial_x \varphi_N\|^2 \sim \|\mathcal{D}_x \varphi_N\|^2 = \|P_{N-1}(\mathcal{D}_x \varphi)\|^2 \leq \|\mathcal{D}_x \varphi\|^2 \sim \|\partial_x \varphi\|^2. \quad (3.26)$$

Integrating from 0 to T gives that

$$\|\partial_x \varphi_N\|_{\Omega} \leq \|\partial_x \varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))} \stackrel{(3.26)}{\lesssim} \|\partial_x \varphi\|_{L^2(0,T;L^2(\mathbb{R}))} = \|\partial_x \varphi\|_{\Omega},$$

since $\varphi \in H_0^1(\mathbb{R})$. \square

THEOREM 3.5. *Let $f \in C^2(\mathbb{R})$ be a nonlinear function such that $f'(0) = 0$, $u_0 \in L^2 \cap L^\infty(\mathbb{R})$. Let u_N be the solution to the spectral approximation (3.2). Assume that u_N is uniformly bounded, i.e. (3.13) holds, and $\frac{1}{\sqrt{N}\epsilon_N} = o(1)$, $m_N \sqrt{\epsilon_N} = o(1)$, as $N \rightarrow \infty$. Then $\{u_N\}$ converges (strongly in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$) to the unique entropy solution, u , of the problem (1.1), where $\Omega \in \mathbb{R} \times [0, T]$ is an open and bounded subset.*

Proof. The uniform boundedness of $\{u_N\}$ in $L^\infty(\mathbb{R} \times [0, T])$ guarantees that there exists a subsequence converges in the weak-* sense of L^∞ , denote this subsequence also $\{u_N\}$ and the limit u . We shall prove that u is the unique entropy solution of (1.1), and the whole sequence $\{u_N\}$ tends to u in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$.

We first show that $\partial_t u_N + \partial_x f(u_N)$ is in a compact set of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$. Let us consider for any $\varphi \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle \partial_t u_N + \partial_x f(u_N), \varphi \rangle_{\Omega} &= \langle \partial_t u_N + \partial_x f(u_N), \varphi - \varphi_N \rangle_{\Omega} + \langle \partial_x (I - P_{N+1})f(u_N), \varphi_N \rangle_{\Omega} \\ &\quad + \langle \partial_t u_N + \partial_x P_{N+1}f(u_N), \varphi_N \rangle_{\Omega} := \sum_{j=1}^3 I_j(\varphi), \end{aligned} \quad (3.27)$$

where $\varphi_N = P_N \varphi \in \mathcal{R}_N$. By Lemma 3.3, the first term can be bounded as

$$|I_1(\varphi)| \leq (\|\partial_t u_N\|_{\Omega} + C\|\partial_x u_N\|_{\Omega})\|\varphi - \varphi_N\|_{\Omega} \lesssim \frac{1}{\sqrt{\epsilon_N}}\|\varphi - \varphi_N\|_{\Omega}.$$

The second term on the right-hand side of (3.27) can be estimated as

$$\begin{aligned} |I_2(\varphi)| &= |\langle (I - P_{N+1})f(u_N), \partial_x \varphi_N \rangle_{\Omega}| \stackrel{(3.24)}{\lesssim} \alpha^{-1} N^{-\frac{1}{2}} \|\partial_x (f(u_N))\|_{\Omega} \|\partial_x \varphi_N\|_{\Omega} \\ &\lesssim \alpha^{-1} N^{-\frac{1}{2}} \|\partial_x u_N\|_{\Omega} \|\partial_x \varphi_N\|_{\Omega} \stackrel{(3.15)}{\lesssim} \frac{1}{\alpha\sqrt{N}\epsilon_N} \|\partial_x \varphi_N\|_{\Omega}. \end{aligned}$$

For the third term on the right-hand side of (3.27), we can obtain:

$$\begin{aligned} |I_3(\varphi)| &\stackrel{(3.17)}{\leq} \epsilon_N |\langle \mathcal{D}_x Q_{m_N} u_N, \mathcal{D}_x Q_{m_N} \varphi_N \rangle_{\Omega}| \leq \epsilon_N \|\mathcal{D}_x Q_{m_N} u_N\|_{\Omega} \|\mathcal{D}_x Q_{m_N} \varphi_N\|_{\Omega} \\ &\stackrel{(3.6), (3.14)}{\lesssim} \sqrt{\epsilon_N} (\|\mathcal{D}_x \varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))} + \alpha m_N \|\varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))}) \\ &\stackrel{(2.17)}{\lesssim} \sqrt{\epsilon_N} (\|\partial_x \varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))} + \alpha m_N \|\varphi_N\|_{L^2(0,T;L^2(\mathbb{R}))}) \end{aligned}$$

Therefore, it follows from (3.24), (3.26) and Lemma 2.1 that

$$\begin{aligned}
& |(\partial_t u_N + \partial_x f(u_N), \varphi)_\Omega| \\
& \lesssim \frac{1}{\alpha\sqrt{N}\epsilon_N} \|\partial_x \varphi\|_\Omega \\
& \quad + \sqrt{\epsilon_N} \left[\|\partial_x \varphi\|_{L^2(0,T;L^2(\mathbb{R}))} + \alpha m_N \left(\|\varphi\|_{L^2(0,T;L^2(\mathbb{R}))} + \frac{1}{\alpha\sqrt{N}} \|\partial_x \varphi\|_{L^2(0,T;L^2(\mathbb{R}))} \right) \right] \\
& \lesssim \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + \sqrt{\epsilon_N} + \frac{m_N \sqrt{\epsilon_N}}{\sqrt{N}} \right) \|\partial_x \varphi\|_\Omega + \sqrt{\epsilon_N} \alpha m_N \|\varphi\|_\Omega \rightarrow 0, \tag{3.28}
\end{aligned}$$

if $N\epsilon_N \rightarrow \infty$ and $m_N \sqrt{\epsilon_N} \rightarrow 0$, as $N \rightarrow \infty$. This gives that $\partial_t u_N + \partial_x f(u_N)$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Let (U, F) be an entropy pair associated to (1.1). Next, we shall show that $\partial_t U(u_N) + \partial_x F(u_N) = (\partial_t u_N + \partial_x f(u_N))U'(u_N)$ is also in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$. We replace the function φ in the above procedure with the function $U'(u_N)\varphi$, then we obtain that for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
& |(\partial_t U(u_N) + \partial_x F(u_N), \varphi)_\Omega| \\
& = |(\partial_t u_N + \partial_x f(u_N), U'(u_N)\varphi)_\Omega| \leq \sum_{j=1}^3 |I_j(U'(u_N)\varphi)| \\
& \stackrel{(3.28)}{\leq} \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + \sqrt{\epsilon_N} + \frac{m_N \sqrt{\epsilon_N}}{\sqrt{N}} \right) \|\partial_x (U'(u_N)\varphi)\|_\Omega + \sqrt{\epsilon_N} \alpha m_N \|U'(u_N)\varphi\|_\Omega \\
& \lesssim \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + \sqrt{\epsilon_N} + \frac{m_N \sqrt{\epsilon_N}}{\sqrt{N}} \right) \|\partial_x u_N\|_\Omega \|\varphi\|_\infty \\
& \quad + \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + \sqrt{\epsilon_N} + \frac{m_N \sqrt{\epsilon_N}}{\sqrt{N}} \right) \|U'(u_N)\|_\infty \|\partial_x \varphi\|_\Omega \\
& \quad + \sqrt{\epsilon_N} \alpha m_N \|U'(u_N)\|_\infty \|\varphi\|_\infty \\
& \lesssim \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + 1 + \frac{m_N}{\sqrt{N}} + \sqrt{\epsilon_N} \alpha m_N \right) \|\varphi\|_\infty + \left(\frac{1}{\alpha\sqrt{N}\epsilon_N} + \sqrt{\epsilon_N} + \frac{m_N \sqrt{\epsilon_N}}{\sqrt{N}} \right) \|\partial_x \varphi\|_\Omega.
\end{aligned}$$

Thus the entropy production $\partial_t U(u_N) + \partial_x F(u_N)$ can be written as a sum of two terms, the first summand is bounded in $L^1(\Omega)$ and the second term tends to 0 in $H_{loc}^{-1}(\Omega)$. Besides, $\partial_t U(u_N) + \partial_x F(u_N)$ is in $W_{loc}^{-1,p}(\mathbb{R} \times (0, T))$ for any $p > 2$, since U and F are continuous and u_N is uniformly bounded in $L^\infty(\mathbb{R} \times (0, T))$. Therefore, in view of the Murat's lemma [5], $\partial_t U(u_N) + \partial_x F(u_N)$ is in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

We conclude that the entropy production of (3.2) is H^{-1} -compact, by compensated compactness arguments [26], implies that u_N converges strongly in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$ to a weak solution, u , of the conservation law (1.1).

It remains to show that u is indeed the unique entropy solution. To this end, we first note that the first two terms on the right-hand side of (3.27) tend to 0, in fact

$$\begin{aligned}
\sum_{j=1}^2 |I_j(U'(u_N)\varphi)| & \lesssim \frac{1}{\alpha\sqrt{N}\epsilon_N} \|\partial_x (U'(u_N)\varphi)\|_\Omega \\
& \leq \frac{1}{\alpha\sqrt{N}\epsilon_N} (\|\partial_x u_N\|_\Omega \|\varphi\|_\infty + \|U'(u_N)\|_\infty \|\partial_x \varphi\|_\Omega) \rightarrow 0,
\end{aligned}$$

if $\sqrt{N}\epsilon_N \rightarrow \infty$. Next, we write the third term in (3.27) as

$$\begin{aligned}
I_3(U'(u_N)\varphi) & = -\epsilon_N \langle \mathcal{D}_x(Q_{m_N} u_N), \mathcal{D}_x P_N(Q_{m_N}(U'(u_N)\varphi)) \rangle_\Omega \\
& = -\epsilon_N \langle \mathcal{D}_x u_N, \mathcal{D}_x P_N(U'(u_N)\varphi) \rangle_\Omega + \epsilon_N \langle \mathcal{D}_x(R_{m_N} u_N), \mathcal{D}_x P_N(U'(u_N)\varphi) \rangle_\Omega \\
& \quad + \epsilon_N \langle \mathcal{D}_x(Q_{m_N} u_N), \mathcal{D}_x P_N(R_{m_N}(U'(u_N)\varphi)) \rangle_\Omega := \sum_{j=1}^3 J_j(\varphi).
\end{aligned}$$

With the fact that for all $\phi, \zeta \in L^2(\mathbb{R})$,

$$\mathcal{D}_x(\phi\zeta) = \mathcal{D}_x\phi\zeta + \phi\partial_x\zeta, \quad (3.29)$$

we can estimate $J_1(\varphi)$ as

$$\begin{aligned} J_1(\varphi) &\stackrel{(3.25)}{=} -\epsilon_N \langle \mathcal{D}_x u_N, P_{N-1}(\mathcal{D}_x(U'(u_N)\varphi)) \rangle_\Omega = -\epsilon_N \langle \mathcal{D}_x u_N, \mathcal{D}_x(U'(u_N)\varphi) \rangle_\Omega \\ &\stackrel{(3.29)}{=} -\epsilon_N \langle \mathcal{D}_x u_N, \mathcal{D}_x(U'(u_N))\varphi \rangle_\Omega - \epsilon_N \langle \mathcal{D}_x u_N, U'(u_N)\partial_x\varphi \rangle_\Omega \\ &= -\epsilon_N \langle \mathcal{D}_x u_N, U''(u_N)\partial_x u_N \varphi \rangle_\Omega - \epsilon_N \langle \mathcal{D}_x u_N, \alpha^2 x U'(u_N)\varphi \rangle_\Omega - \epsilon_N \langle \mathcal{D}_x u_N, U'(u_N)\partial_x\varphi \rangle_\Omega \\ &= -\epsilon_N \langle \partial_x u_N, U''(u_N)\partial_x u_N \varphi \rangle_\Omega - \epsilon_N \langle \alpha^2 x u_N, U''(u_N)\partial_x u_N \varphi \rangle_\Omega - \epsilon_N \langle \mathcal{D}_x u_N, U'(u_N)\mathcal{D}_x\varphi \rangle_\Omega \\ &\lesssim \alpha^2 \epsilon_N \|\varphi\|_\infty \langle x u_N, \partial_x u_N \rangle_\Omega + \epsilon_N \|\mathcal{D}_x u_N\|_\Omega \|\mathcal{D}_x\varphi\|_\Omega \lesssim \alpha^2 \epsilon_N \|\varphi\|_\infty \|\partial_x u_N\|_\Omega + \sqrt{\epsilon_N} \|\mathcal{D}_x\varphi\|_\Omega \\ &\lesssim \sqrt{\epsilon_N} (\alpha^2 \|\varphi\|_\infty + \|\partial_x\varphi\|_\Omega) \rightarrow 0, \end{aligned}$$

due to Remark 2.1 and the convexity of U . And the other two terms also tend to 0,

$$\begin{aligned} |J_2(\varphi)| &\stackrel{(3.25)}{=} \epsilon_N |\langle \mathcal{D}_x(R_{m_N} u_N), P_{N-1} \mathcal{D}_x(U'(u_N)\varphi) \rangle_\Omega| \stackrel{(2.10)}{=} \epsilon_N |\langle \mathcal{D}_x(R_{m_N} u_N), \mathcal{D}_x(U'(u_N)\varphi) \rangle_\Omega| \\ &\leq \epsilon_N \|\mathcal{D}_x(R_{m_N} u_N)\|_\Omega \cdot \|\mathcal{D}_x(U'(u_N)\varphi)\|_\Omega \stackrel{(3.12)}{\lesssim} \epsilon_N \alpha m_N \|u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \cdot \|\mathcal{D}_x\varphi\|_\Omega \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} |J_3(\varphi)| &\stackrel{(3.25),(2.10)}{=} \epsilon_N \langle \mathcal{D}_x(Q_{m_N} u_N), \mathcal{D}_x(R_{m_N}(U'(u_N)\varphi)) \rangle_\Omega \\ &\leq \epsilon_N \|\mathcal{D}_x(Q_{m_N} u_N)\|_\Omega \cdot \|\mathcal{D}_x(R_{m_N}(U'(u_N)\varphi))\|_\Omega \\ &\stackrel{(3.14),(3.12)}{\lesssim} \alpha m_N \sqrt{\epsilon_N} \|U'(u_N)\varphi\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \alpha m_N \sqrt{\epsilon_N} \|U'(u_N)\|_\infty \|\varphi\|_{L^2(0,T;L^2(\mathbb{R}))} \rightarrow 0. \end{aligned}$$

Thus, we arrive at the convergence theorem. \square

REMARK 3.2. *The conditions $\frac{1}{\sqrt{N}\epsilon_N} = o(1)$, $m_N \sqrt{\epsilon_N} = o(1)$, as $N \rightarrow \infty$ are satisfied if the conditions in Theorem 2 [1] hold, that is $m_N = \mathcal{O}(N^\beta)$, $\epsilon_N = \mathcal{O}(N^{-\theta})$, with $0 < 2\beta < \theta < \frac{1}{2}$.*

4. Hermite spectral filters. This method is based on trying to localize the information that determines the Fourier-Hermite coefficients.

Suppose that $f(x)$ is a C^∞ function at $x \in \mathbb{R}$ except for one point of irregularity. Suppose also that $f(x)$ has the following expansion in terms of the Hermite functions:

$$f(x) = \sum_{k=0}^{\infty} a_k H_k^\alpha(x).$$

And denote the first N -term partial sum to be

$$f_N(x) = \sum_{k=0}^N a_k H_k^\alpha(x).$$

Even for large N , $f_N(x)$ is oscillatory. Let y be a point such that $f(x) \in C^\infty([y - x_K \epsilon, y + x_K \epsilon])$, where x_K is the largest root of H_K^α . The spectral filter is defined as

$$\mathcal{F}_{\Psi_K}[f](y) = \int_{\mathbb{R}} f_N(x) \Psi_K(x, y) dx, \quad (4.1)$$

where the localization kernel $\Psi_K(x, y)$ is

$$\Psi_K(x, y) = \Psi_K(\xi) = \begin{cases} \frac{C_K}{\epsilon} \sum_{k=0}^K H_k^\alpha(0) H_k^\alpha(\xi), & |\xi| < x_K - \delta, \xi = \frac{x-y}{\epsilon} \\ \text{smoothly connected to 0,} & 1 - \delta \leq |\xi| < x_K \\ 0, & |\xi| \geq x_K \end{cases}, \quad (4.2)$$

where $\delta \ll 1$, x_K denotes the largest root of $H_K^\alpha(x)$, $K > 0$, and C_K denotes the normalization constant such that

$$C_K \sum_{k=0}^K H_k^\alpha(0) \int_{-x_K}^{x_K} H_k^\alpha(\xi) d\xi = 1.$$

PROPOSITION 4.1. *Let $\Psi(x, y)$ be the Kernel function defined in (4.2), and let y be a point such that $f(x) \in C^\infty([y - x_K \epsilon, y + x_K \epsilon])$, for some small $\epsilon > 0$. Then*

$$|\mathcal{F}_{\Psi_K}[f](y) - f(y)| \lesssim N^{-\frac{p}{2}} \|\partial_x^p \Psi_K\| + \sqrt{C_K} \epsilon^{2K} K^{-K-\frac{1}{4}} \|\partial_x^{2K} f\|_{L^1}.$$

Proof. It is clear that

$$\mathcal{F}_{\Psi_K}[f](y) = \int_{\mathbb{R}} f_N(x) \Psi_K(x, y) dx = \int_{\mathbb{R}} f(x) \Psi_K(x, y) dx + \int_{\mathbb{R}} [f_N(x) - f(x)] \Psi_K(x, y) dx. \quad (4.3)$$

Let us denote $\Psi_{K,N}(x, y) = P_{N,x} \Psi_K(x, y)$, where $P_{N,x}$ is the projection operator with respect to x onto the linear subspace \mathcal{R}_N defined in (2.16). The second term on the right-hand side of (4.3) can be estimated as

$$\left| \int_{\mathbb{R}} [f_N(x) - f(x)] \Psi_K(x, y) dx \right| = \left| \int_{\mathbb{R}} [f_N(x) - f(x)] \cdot [\Psi_K(x, y) - \Psi_{K,N}(x, y)] dx \right| \quad (4.4)$$

$$\leq \|f_N(x) - f(x)\| \cdot \|\Psi_K - \Psi_{K,N}\|(y) \lesssim N^{-\frac{p}{2}} \|\partial_x^p \Psi_K\|, \quad (4.5)$$

where (4.4) follows from the fact that $\Psi_{K,N}(x, y)$ is orthogonal to all $H_n^\alpha(x)$, with $n \geq N + 1$, and (4.5) is a direct application of the approximation theory for smooth functions. Moreover,

$$\begin{aligned} \int_{\mathbb{R}} f(x) \Psi_K(x, y) dx &\stackrel{(4.2)}{=} \frac{C_K}{\epsilon} \int_{\mathbb{R}} f(x) \sum_{k=0}^K H_k^\alpha(0) H_k^\alpha(\xi) dx = C_K \int_{-x_K}^{x_K} f(\epsilon\xi + y) \sum_{k=0}^K H_k^\alpha(0) H_k^\alpha(\xi) d\xi \\ &= \sum_{k=0}^K \bar{H}_k^\alpha(0) \int_{-x_K}^{x_K} f(\epsilon\xi + y) \bar{H}_k^\alpha(\xi) d\xi, \end{aligned}$$

where $\bar{H}_k^\alpha(x) = \sqrt{C_K} H_k^\alpha(x)$. Let $g(\xi) = f(\epsilon\xi + y)$. The above equality can be written as

$$\begin{aligned} \int_{\mathbb{R}} f(x) \Psi_K(x, y) dx &\sim \sum_{k=0}^K \hat{g}_k \bar{H}_k^\alpha(0) = \sum_{k=0}^{\infty} \hat{g}_k \bar{H}_k^\alpha(0) - \sum_{k=K+1}^{\infty} \hat{g}_k \bar{H}_k^\alpha(0) \stackrel{(2.3)}{=} g(0) - \sum_{k=\lceil \frac{K+1}{2} \rceil}^{\infty} \hat{g}_{2k} \bar{H}_{2k}^\alpha(0) \\ &= f(y) - \sum_{k=\lceil \frac{K+1}{2} \rceil}^{\infty} \hat{g}_{2k} \bar{H}_{2k}^\alpha(0), \end{aligned} \quad (4.6)$$

where \hat{g}_k is the Fourier-Hermite coefficients under the orthogonal basis $\{\bar{H}_k^\alpha\}_{k=0}^{\infty}$ and $\lceil \cdot \rceil$ is the ceiling function of \cdot . Let us recall that the decay rate of the Fourier-Hermite coefficients \hat{g}_{2k} , $k \gg 1$, satisfies

$$|\hat{g}_{2k}| \leq k^{-m} \|\partial_x^{2m} g\|_{L^1},$$

where $H^{2m,1}(\mathbb{R})$ is the Sobolev space in which all the derivatives of the functions upto the order $2m$ are integrable. Therefore, we have

$$\left| \sum_{k=\lceil \frac{K+1}{2} \rceil}^{\infty} \hat{g}_{2k} \bar{H}_{2k}^\alpha(0) \right| \stackrel{(2.3)}{\lesssim} \sqrt{C_K} K^{-K-\frac{1}{4}} \|\partial_x^{2K} g\|_{L^1}. \quad (4.7)$$

Combining (4.3) and (4.5)-(4.7), we obtain that

$$|\mathcal{F}_{\Psi}[f](y) - f(y)| \lesssim N^{-\frac{p}{2}} \|\partial_x^p \Psi\| + \sqrt{C_K} \epsilon^{2K} K^{-K-\frac{1}{4}} \|\partial_x^{2K} f\|_{L^1}.$$

□

5. Numerical results. In this section we present the numerical simulations of our Hermite spectral viscosity method (3.2) to the inviscid Burger's equation

$$\partial_t u + \frac{1}{2} \partial_x (u^2) = 0, \quad (5.1)$$

in \mathbb{R} , with the initial condition $u(x, 0) = e^{-x^2}$. We compute the same problem in [1] for the purpose of comparison. The exact solution is given implicitly by the method of characteristics, i.e.,

$$u(\eta + te^{-\eta^2}, t) = e^{-\eta^2}. \quad (5.2)$$

And the shock presents at time $T^* = \left(\frac{\epsilon}{2}\right)^{\frac{1}{2}} \approx 1.1658$. All of the numerical results present in this section are at time $t = 1.5 > T^*$.

In our numerical simulations, we implement the scheme (3.2) and overcome the difficulty of accurately approximating the Fourier-Hermite coefficients (mentioned in section 6 [1]) with the help of the suitable scaling factor α . The optimal choice of the scaling factor to accurately resolve the functions is still open, let alone the solution to some partial differential equations. But the suitable choice of the scaling factor is investigated in [25], [2], [3], [18], etc. It is known so far that the scaling factor should match the asymptotical behavior of the function/solution to be resolved, refer to [18] for details. From (5.2), for any fixed time t , it is easy to see that $u(x, t) \sim e^{-x^2}$, as $|x| \gg 1$. According to the rules in [18], we choose $\alpha = \sqrt{2}$ so that the generalized Hermite functions can resolve the solution well.

Let $\varphi = H_m^\alpha(x)$, $m = 0, 1, \dots, N$, in (3.2). The key component of numerically approximating the second term in (3.2)

$$\langle \partial_x (P_{N+1} u_N^2), \varphi \rangle = -\langle u_N^2, \partial_x H_m^\alpha(x) \rangle$$

is to use the Gauss-Hermite quadrature rule to compute the integral $\int_{\mathbb{R}} H_l^\alpha(x) H_n^\alpha(x) H_m^\alpha(x) dx$, $l, n, m = 0, 1, \dots, N$. This term yields a $(N+1)$ -vector of combinations of $\tilde{u}_l \tilde{u}_n$, $l, n = 0, 1, \dots, N$. And the third term gives a $(N+1) \times (N+1)$ diagonal triangle (by (3.17)) multiplying the vector \tilde{u} . Therefore, the coefficients $\tilde{u}_m(t)$, $m = 0, \dots, N$, are the solution of a nonlinear system of ordinary differential equation. It is solved by using the fourth order Runge-Kutta method with an adaptive time step (*ode45* in Matlab).

In Figure 5.1, we show the result of the spectral approximation (3.2) without viscosity ($\epsilon_N = 0$) for $N = 15$ and $N = 40$. The Gibbs phenomenon prevent the convergence of our method, even in the intervals where the solution is smooth. The larger N is, the better resolution at the point of discontinuity we achieve, but the oscillations seem to be more wild with the increase of N .

REMARK 5.3. *Compared with the pseudospectral viscosity method based on the Hermite functions introduced in [1] (see Figure 6.1-6.3), we can resolve the discontinuity well even with much smaller N .*

The viscosity introduced in the spectral approximation (3.2) depends on the parameters ϵ_N , m_N and the operator Q_{m_N} . Let us try the following multipliers \hat{q}_k^i , $i = 1, 2, 3$, as those used in [1]:

$$\hat{q}_k^1 = \frac{N}{N - m_N} \left(1 - \frac{m_N}{k}\right); \quad (5.3)$$

$$\hat{q}_k^2 = \frac{k - m_N}{N - m_N}; \quad (5.4)$$

$$\hat{q}_k^3 = \exp \left[- \left(\frac{k - N}{k - m_N} \right)^2 \right], \quad (5.5)$$

for $k > m_N$.

In Figure 5.2, we add the viscosity by suitably tuning the parameters $\epsilon_N = 0.5N^{-0.33}$ and $m_N = 5N^{0.16}$. We plot the results for $N = 40$ and $t = 1.5$ with different operators Q_{m_N} given by (5.3)-(5.5). The conditions $m_N = \mathcal{O}(N^\beta)$, $\epsilon_N = \mathcal{O}(N^{-\theta})$, with $0 < 2\beta < \theta < \frac{1}{2}$, are satisfied by

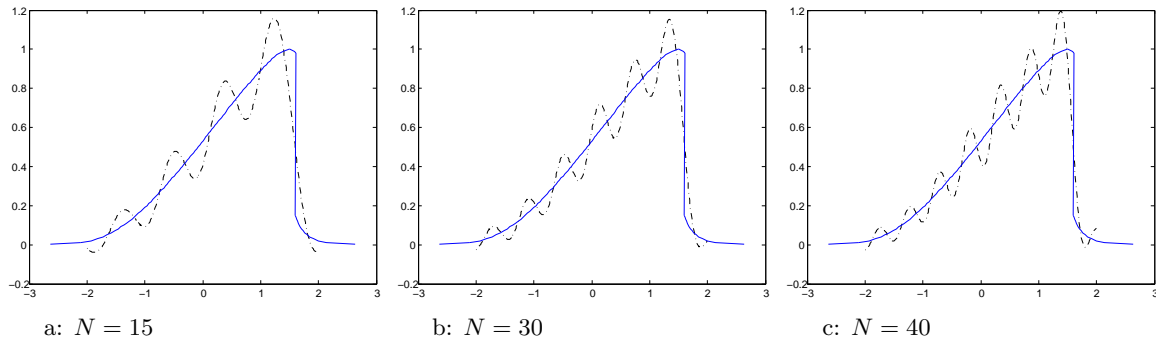


Fig. 5.1 Graphs of the exact solution of Burger's equation (solid line) and of the spectral approximation without viscosity (dashed line), with $N = 15$, $N = 30$ and $N = 40$.

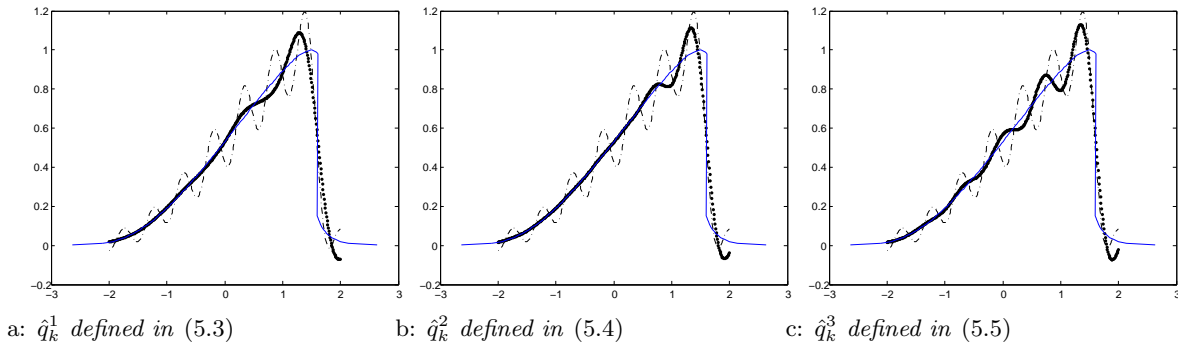


Fig. 5.2 Graphs of the exact solution of inviscid Burger's equation (solid line) and of the spectral approximation without viscosity (dashed line) with $N = 40$. The spectral viscosity method with $\epsilon_N = 0.5N^{-0.33}$, $m_N = 5N^{0.16}$ and different operators Q_{m_N} are plotted in dotted line.

all the above choice of q_k^i , $i = 1, 2, 3$. Hence, the convergence of the spectral viscosity method is guaranteed.

We use the spectral filter developed in section 4 to improve the resolution of the vanishing viscosity approximation away from the discontinuity. In Figure 5.3, we plot the approximate solution before and after filtering with various viscosity operators \hat{q}_k^i , $i = 1, 2, 3$. We choose the localization parameter $\epsilon = 0.4$ and $K = N = 40$. It is clear to see that the postprocessed solution gives a clear improvement of the convergence rate at points away from the discontinuity. However, the resolution near the point of discontinuity becomes worse. Therefore, better filters to resolve the discontinuity are imperative.

6. Conclusion. In this paper, we propose a new spectral viscosity method based on the generalized Hermite functions for the solution of nonlinear scalar conservation laws in the whole line. Our scheme is superior to the existing Hermite spectral viscosity method proposed in [1] in two folds. On the one hand, we remove a restrictive condition in the proof of the convergence theorem of our scheme; on the other hand, our scheme is implementable. The fewer oscillations and better resolutions are obtained with much smaller truncation modes N , even before adding the viscosity term. Our scheme has been shown rigorously to converge to the unique entropy solution by using compensated compactness arguments. Moreover, we propose a new spectral filter, the postprocessing method, of the approximate solution obtained by our method to eliminate the oscillation away from the discontinuity. The spectral accuracy has been recovered at the points away from the discontinuity after using the vanishing viscosity and the spectral filter. However, recovering the spectral

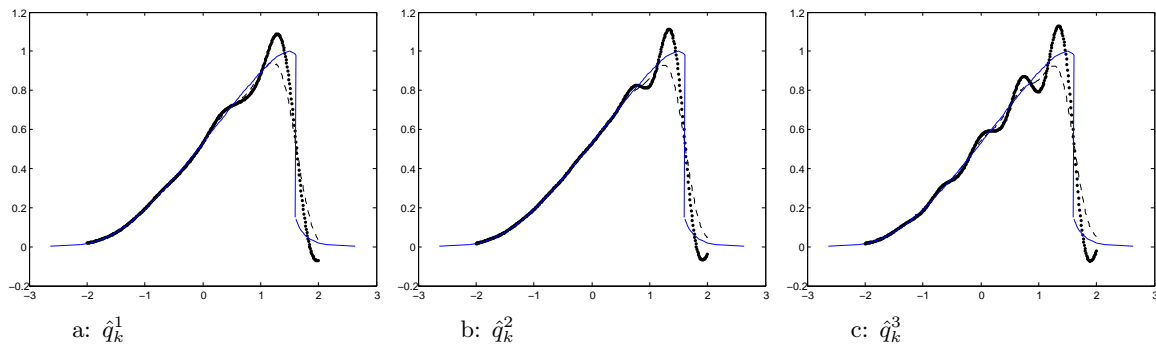


Fig. 5.3 Graphs of the exact solution of inviscid Burger's equation (solid line) and of the spectral approximation (dotted line) with various viscosity and $N = 40$. The filtered solutions are plotted in dashed line.

accuracy near the point of discontinuity is still open, and is imperative to study further to derive an analogue to the Gegenbauer partial sum in the periodic piecewise functions' case.

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