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Local edge detectors using a sigmoidal transformation for piecewise smooth data

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ABSTRACT

For piecewise smooth data, edges can be recognized by jump discontinuities in the data. Successful edge detection is essential in digital signal processing as the most relevant information is often observed near the edges in each segmented region. In this paper, using the concentration property of existing local edge detectors and the clustering property of sigmoidal transformations, we provide enhanced edge detectors which diminish the oscillations of the local detector near jump discontinuities as well as highly improve rate of convergence away from the discontinuities. Numerical results of some examples illustrate efficiency of the presented method.

1. Introduction

Detection of edges is one of fundamental and important tools in digital signal (image) processing, pattern recognition, and many other scientific applications. It aims at determining boundaries of each particular region identified by jump discontinuities. There are lots of prominent edge detection methods [1–13]. If the edge detection is performed successfully, the amount of the signal data to be processed may be significantly reduced and thus redundant information is filtered out while principal properties of the data are preserved. As a result subsequent process of analyzing the original signal becomes rather simplified. However, edge detection is not always successful because real signals are moderately complex, in general. Edges extracted from the complex signals (especially, images) are often hampered by fragmentation, missing edge segments, and false edges [12]. Moreover, applying a threshold to decide whether an edge is present or not at a grid point is not a simple problem because jump discontinuities occur at every grid point in discrete data.

In the literature [4–6], a family of edge detectors associated with particular concentration factors was provided. Though the detectors can effectively find edges from spectral information, there are two major drawbacks as mentioned in [7]:

1. In order to pinpoint the edges one has to employ an outside threshold parameter to quantify the jump discontinuities of large magnitude.
2. Oscillations, depending on the associated concentration factors, appear near the jump discontinuities.

In the work [7], an adaptive edge detection method based on a nonlinear limiter was proposed to suppress unwanted oscillations near jump discontinuities. Nevertheless, the problems mentioned above do not seem to be completely resolved. In this paper we aim to develop enhanced local edge detectors, including a simple global threshold parameter, which can eliminate the unwanted oscillations near jump discontinuities as well as further improve convergence away from the discontinuities.

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This work is organized as follows: In Section 2 we recall existing local edge detectors and the sigmoidal transformation with its basic properties which play important role in developing a new method. In Section 3, using the concentration property of the local edge detectors and the clustering property of the sigmoidal transformation, we develop enhanced local edge detectors without need of any outside threshold parameter. Finally, to show efficiency of the presented method, some numerical examples are explored in Section 4.

2. The local edge detector and sigmoidal transformations

Let $f(x),\ -\pi \le x < \pi$, be a 2π periodic piecewise smooth function for which we wish to identify the points of discontinuity. We denote by $[f](x)$ a jump function defined as

$$[f](x) := f(x+) - f(x-),$$

where $f(x\pm)$ indicate the right-hand and left-hand side limits of $f(x)$ at x. Suppose $2N+1$ values, $f_j = f(x_j)$ are given for the grid points $x_j = -\pi + h\cdot j,\ j = 0, 1, \ldots, 2N$, where $h = 2\pi/(2N+1)$. We denote by d a jump discontinuity of $f(x)$ such that $[f](d) \neq 0$.

In the literature [7] a local edge detector based on the difference formulas was introduced as

$$T_{2p+1,h}(x) := \binom{2p}{p}^{-1} \Delta^{2p+1} f_j, \quad x_j \le x < x_{j+1} \tag{1}$$

for each j, where $\Delta^{2p+1} f_j$ indicates the difference formula of order $2p+1$ for an integer $p \ge 0$. It has a so-called *concentration property* for the jump discontinuity d as follows.

$$T_{2p+1,h}(x) = \begin{cases} (-1)^l \dfrac{Q_{l,p}}{Q_{0,p}} [f](d) + O(h), & \text{if } x_{j-l} \le d < x_{j+1-l},\ |l| \le p \\ O\left(h^{2p+1}\right), & \text{otherwise} \end{cases} \tag{2}$$

as $h \to 0$ (or $N \to \infty$), where $Q_{l,p} = \binom{2p}{p+|l|}$. From (2) one can see the oscillatory behavior of $T_{2p+1,h}(x)$ near the discontinuity d is increasing as p grows.

Because of the oscillations near the jump discontinuities and the various orders of convergence away from the discontinuities, the aforementioned edge detector needs to be further improved. For example, based on a threshold parameter signifying the minimal amplitude below which jump discontinuities are neglected, the edge detector $T_{2p+1,h}(x)$ can be enhanced by separating the smooth regions from the neighborhoods of the jump discontinuities (see [5]). However, the need of an outside threshold parameter in the enhancement procedure is an impediment for detecting edges.

On the other hand we introduce a sigmoidal transformation $\gamma_m(t),\ 0 \le t \le 1$, of order $m \ge 1$ which satisfies the following properties.

(a) $\gamma_m(t) \in \mathbf{C}^1[0,1] \cap \mathbf{C}^\infty(0,1)$
(b) $\gamma_m(t) + \gamma_m(1-t) = 1,\ 0 \le t \le 1$, with $\gamma_m(0) = 0$
(c) $\gamma_m(t)$ and $\gamma_m'(t)$ are strictly increasing on $[0, 1]$ and $[0, 1/2]$, respectively
(d) $\gamma_m^{(j)}(t) = O\left(t^{m-j}\right)$ near $t = 0, j = 0, 1, \ldots, m$.

The property (b) implies that

$$\gamma_m(s) = s, \quad s = 0, \frac{1}{2}, 1 \tag{3}$$

and, from (b) and (d), it follows that near $t = 1$

$$\gamma_m^{(j)}(t) = \delta_{0,j} + O\left((1-t)^{m-j}\right), \quad j = 0, 1, \ldots, m, \tag{4}$$

where $\delta_{0,j}$ is Kronecker's delta. Particularly, we can notice that $\gamma_m(x)$ satisfies

$$\gamma_m(t) = \begin{cases} O\left(t^m\right), & t < \dfrac{1}{2} \\ 1 + O\left((1-t)^m\right), & t > \dfrac{1}{2} \end{cases} \tag{5}$$

as $m \to \infty$. That is, every point on the intervals $[0, 1/2)$ and $(1/2, 1]$ is respectively clustered to the values 0 and 1 by $\gamma_m(t)$ for m large enough, which we call a *clustering property*. For example, Fig. 1 illustrates the aforementioned properties of a simple sigmoidal transformation $\gamma_m(x)$ defined by Prössdorf [14] as follows.

$$\gamma_m(t) = \frac{t^m}{t^m + (1-t)^m}, \quad 0 \le t \le 1 \tag{6}$$

which we take in numerical implementation later.

Originally, sigmoidal transformations are used for numerical evaluation of singular integrals (see [15–18]). However, in this paper sigmoidal transformations having the property (5) play prominent role in constructing new edge detectors.

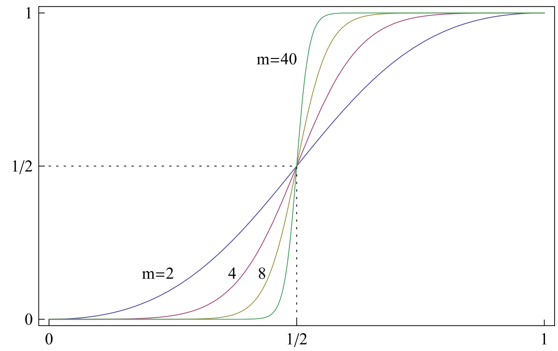


Fig. 1. Behavior of the sigmoidal transformation $\gamma_m(x)$ for $m = 2, 4, 8, 40$.

3. Enhancement of the local edge detector

First, for a 2π periodic piecewise smooth function $f(x)$ and for the grid points x_j's defined in Section 2, we consider the first order local edge detector

$$T_{1,h}(x) = f_{j+1} - f_j, \quad x_j \le x < x_{j+1} \tag{7}$$

for each $j = 0, 1, \ldots, 2N$. A jump discontinuity d is identified by the adjacent points x_k and x_{k+1} such that $x_k \le d < x_{k+1}$.

In order to improve the rate of convergence of the detector $T_{1,h}(x)$ away from the discontinuity, we set a function

$$G_{1,h}^{[m]}(x) := T_{1,h}(x)\gamma_m\left(\theta_1(x)\right) \tag{8}$$

where $\gamma_m(x)$ is a sigmoidal transformation of order m and $\theta_1(x)$ is a function defined as

$$\theta_1(x) = \frac{\left|T_{1,h}(x)\right|}{\left|T_{1,h}(x)\right| + \left|T_{1,h}(x-h)\right| + \left|T_{1,h}(x+h)\right| + \delta} \tag{9}$$

for $\delta > 0$ and $h = x_{j+1} - x_j = 2\pi/(2N+1)$.

Theorem 1. *The first order edge detector $G_{1,h}^{[m]}(x)$, with m large enough, has the following asymptotic behavior: For all $x_j \le x < x_{j+1}$*

$$G_{1,h}^{[m]}(x) = \begin{cases} [f](d) + O\left(\dfrac{1}{2^m}\right), & \text{if } d \in [x_j, x_{j+1}) \text{ and } |[f](d)| > \delta \\ O\left(\dfrac{1}{2^m}\right), & \text{if } d \in [x_j, x_{j+1}) \text{ and } |[f](d)| < \delta \\ O\left(h^{m+1}\right), & \text{otherwise} \end{cases} \tag{10}$$

as $h \to 0$.

Proof. The concentration property (2) implies that

$$T_{1,h}(x) = \begin{cases} [f](d) + O(h), & \text{if } d \in [x_j, x_{j+1}) \\ O(h), & \text{otherwise} \end{cases}$$

as $h \to 0$. Suppose that $d \in [x_j, x_{j+1})$. Then from the concentration property of $T_{1,h}(x)$ above we have

$$\theta_1(x) = \frac{|[f](d)| + O(h)}{|[f](d)| + O(h) + \delta}$$

for $x_j \le x < x_{j+1}$, as $h \to 0$. Thus

$$\theta_1(x) \gtrless \frac{1}{2} \quad \text{if } |[f](d)| \gtrless \delta, \text{ respectively.}$$

On the other hand, when $d \notin [x_j, x_{j+1})$,

$$\theta_1(x) = O(h).$$

Therefore, from the definition of $G_{1,h}^{[m]}(x)$ and the clustering property of $\gamma_m(x)$ in (5) we have the asymptotic relations in (10). □

Referring to the results in Theorem 1, we note the following.

i. $G_{1,h}^{[m]}(x)$ with large m can highly improve convergence rate of $T_{1,h}(x)$ away from the jump discontinuity. Thus $G_{1,h}^{[m]}(x)$ is available for an edge detector which enhances separation of the jump discontinuity from smooth regions.
ii. We can use δ for a global threshold parameter in the sense that every jump discontinuity whose amplitude is below δ is neglected. In addition, δ prevents the denominator of $\theta_1(x)$ vanishing.
iii. Thanks to the terms $|T_{1,h}(x-h)|$ and $|T_{1,h}(x+h)|$ in (9), even the badly fluctuating part in the smooth region can be flatted as illustrated by the numerical example for $f_2(x)$ in the last section.

Similarly to $G_{1,h}^{[m]}(x)$, we define new higher order local edge detectors as follows.

$$G_{2p+1,h}^{[m]}(x) := T_{2p+1,h}(x)\gamma_m\left(\theta_{2p+1}(x)\right), \tag{11}$$

where $p \geq 1$. The function $\theta_{2p+1}(x)$ is defined as

$$\theta_{2p+1}(x) = \frac{\left|T_{2p+1,h}(x)\right|}{\left|T_{2p+1,h}(x)\right| + \sum_{k=1}^{p}\left\{|T_{1,h}(x-kh)| + |T_{1,h}(x+kh)|\right\} + \delta} \tag{12}$$

with $\delta > 0$. Then we have

Theorem 2. *The higher order edge detector $G_{2p+1,h}^{[m]}(x)$, with m large enough, has the following asymptotic behavior: For all $x_j \leq x < x_{j+1}$*

$$G_{2p+1,h}^{[m]}(x) = \begin{cases} [f](d) + O\left(\frac{1}{2^m}\right), & \text{if } d \in [x_j, x_{j+1}) \text{ and } |[f](d)| > \delta \\ O\left(\frac{1}{2^m}\right), & \text{if } d \in [x_j, x_{j+1}) \text{ and } |[f](d)| < \delta \\ O\left(\frac{1}{2^m}\right), & \text{if } d \in [x_{j-p}, x_j) \cup [x_{j+1}, x_{j+1+p}) \\ O\left(h^{(2p+1)(m+1)}\right), & \text{otherwise} \end{cases} \tag{13}$$

as $h \to 0$.

Proof. The proof for each case except for $d \in [x_{j-p}, x_j) \cup [x_{j+1},\ x_{j+1+p})$ is similar to that of Theorem 1. Thus we consider the exceptional case only as follows.

Suppose that $d \in [x_{j+\alpha}, x_{j+\alpha+1})$ for some integer α such that $1 \leq |\alpha| \leq p$. Then from the definition of $\theta_{2p+1}(x)$ in (12) and the concentration property of $T_{2p+1,h}(x)$ in (2) we have

$$\begin{aligned}\theta_{2p+1}(x) &\leq \frac{\left|T_{2p+1,h}(x)\right|}{\left|T_{2p+1,h}(x)\right| + \left|T_{1,h}(x+\alpha h)\right|} \\ &= \frac{|c_\alpha[f](d)| + O(h)}{|c_\alpha[f](d)| + O(h) + |[f](d)|} < \frac{1}{2}\end{aligned}$$

as $h \to 0$, where $c_\alpha = (-1)^\alpha \frac{Q_{\alpha,p}}{Q_{0,p}}$. The last inequality holds by the fact $|c_\alpha| < 1$. Therefore, the definition of $G_{2p+1,h}^{[m]}(x)$ and the clustering property of $\gamma_m(x)$ in (5) imply that

$$G_{2p+1,h}^{[m]}(x) = O\left(\frac{1}{2^m}\right).$$

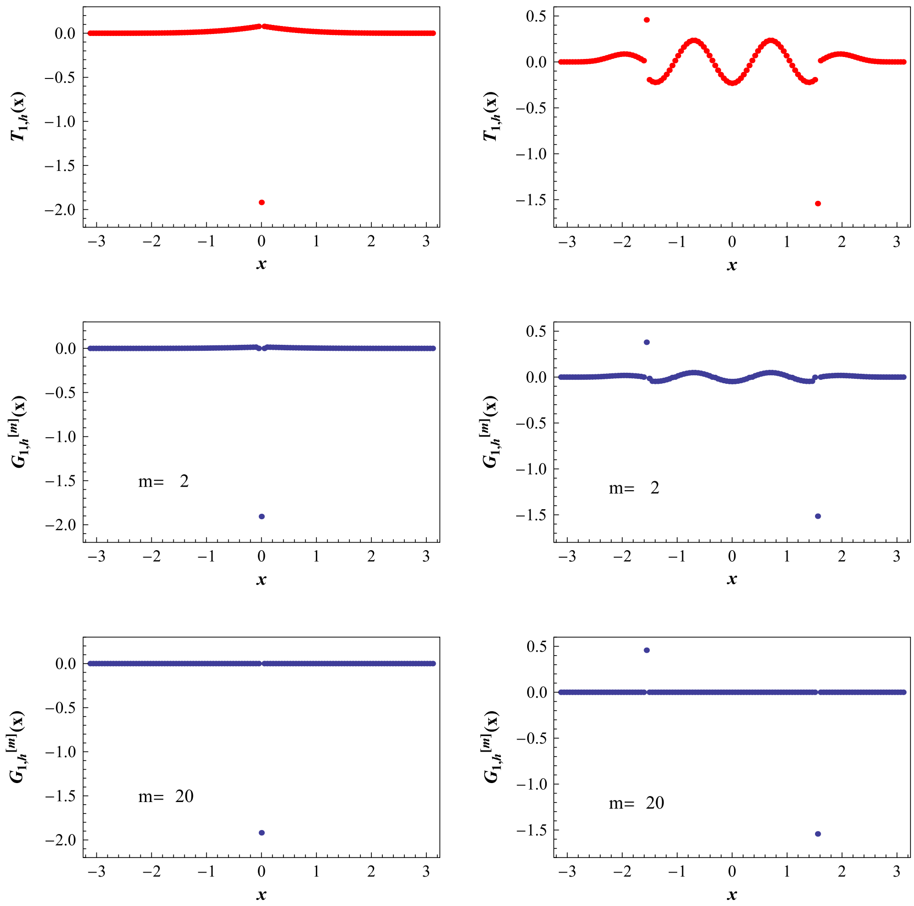
This completes the proof. □

The theorem above indicates that the higher order edge detector $G_{2p+1,h}^{[m]}(x)$ with large m can sufficiently enhance the separation of the neighborhood of the jump discontinuity d from the smooth regions without any oscillation.

4. Examples

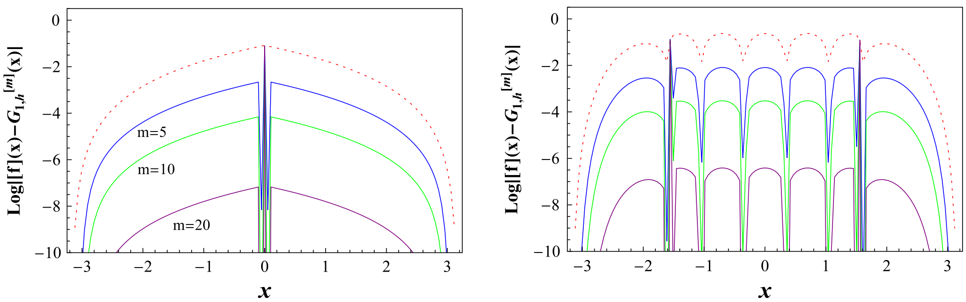
We take the following examples employed by Gelb and Tadmor [7].

$$f_1(x) = \begin{cases} \left(\frac{x+\pi}{\pi}\right)^5, & -\pi \leq x < 0 \\ \left(\frac{x-\pi}{\pi}\right)^5, & 0 < x < \pi \end{cases} \tag{14}$$



(a) $f_1(x)$. (b) $f_2(x)$.

Fig. 2. The first order local edge detector $T_{1,h}(x)$ and the presented edge detector $G^{[m]}_{1,h}(x)$ with $m = 2$ and 20 in (8) for the data of $f_1(x)$ and $f_2(x)$ with $N = 60$.



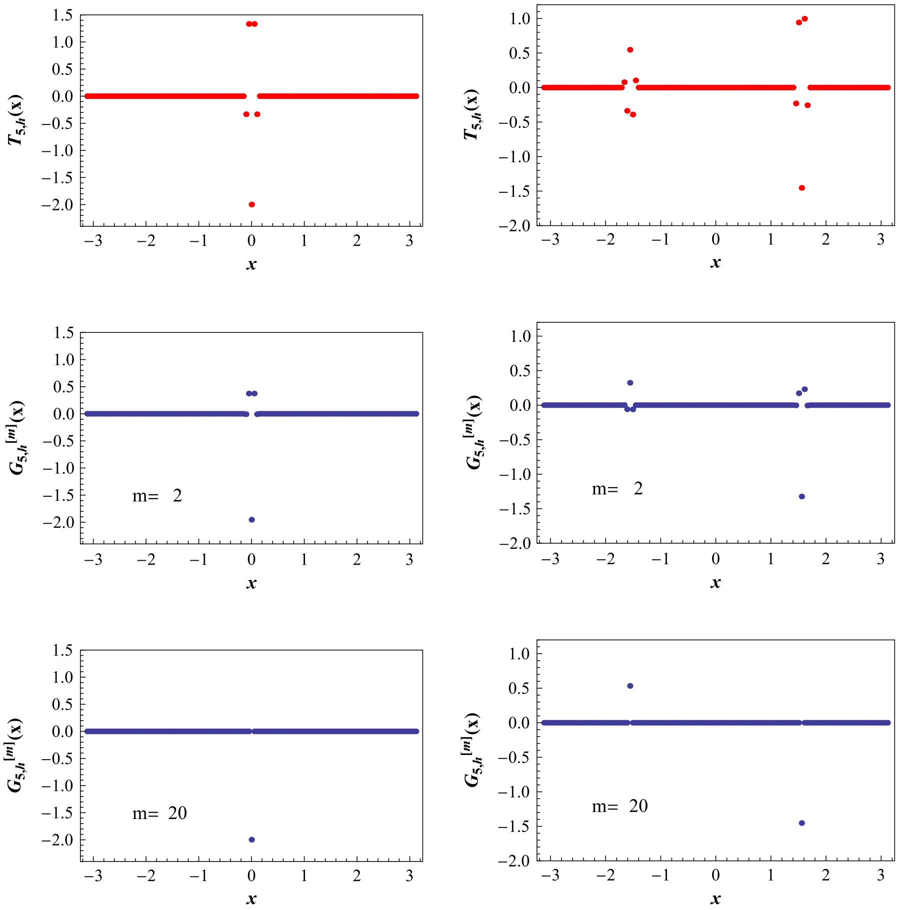
(a) $f_1(x)$. (b) $f_2(x)$.

Fig. 3. Log errors of $T_{1,h}(x)$ (dotted line) and those of $G^{[m]}_{1,h}(x)$ (solid lines) with $m = 5, 10, 20$ for the data of $f_1(x)$ and $f_2(x)$ with $N = 60$.

whose jump function is

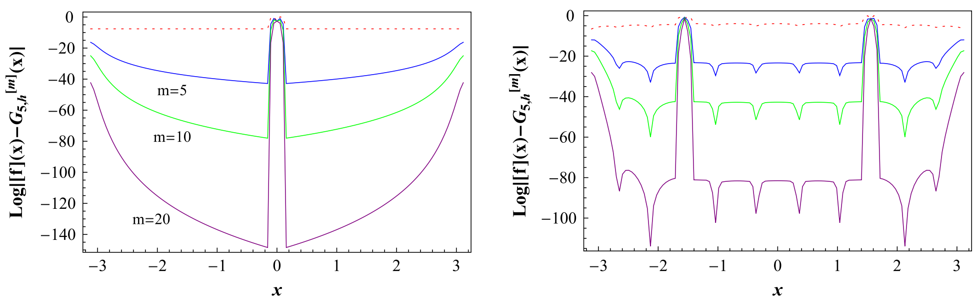
$$[f_1](x) = \begin{cases} -2, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and



(a) $f_1(x)$. (b) $f_2(x)$.

Fig. 4. The fifth order local edge detector $T_{5,h}(x)$ and the presented edge detector $G_{5,h}^{[m]}(x)$ with $m = 2$ and 20 in (11) for the data of $f_1(x)$ and $f_2(x)$ with $N = 60$.



(a) $f_1(x)$. (b) $f_2(x)$.

Fig. 5. Log errors of $T_{5,h}(x)$ (dotted line) and those of $G_{5,h}^{[m]}(x)$ (solid lines) with $m = 5, 10, 20$ for the data of $f_1(x)$ and $f_2(x)$ with $N = 60$.

$$f_2(x) = \begin{cases} \sin^7(x+\pi), & -\pi \le x < -\dfrac{\pi}{2} \\ \left(\dfrac{x}{\pi}\right)^3 - \sin\left(\dfrac{9x}{2}\right) + 1, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \sin^7(x-\pi), & \dfrac{\pi}{2} < x < \pi \end{cases} \tag{15}$$

whose jump function is

$$[f_2](x) = \begin{cases} 0.5821, & x = -\dfrac{\pi}{2} \\ -1.418, & x = \dfrac{\pi}{2} \\ 0, & x \neq \pm\dfrac{\pi}{2}. \end{cases}$$

Fig. 2 shows that the presented first order edge detector $G^{[m]}_{1,h}(x)$, with $\delta = 10^{-6}$ in (9), enhances separation of the jump discontinuities from smooth regions. It should be noted that $G^{[m]}_{1,h}(x)$ flats sufficiently both the vicinity of each jump discontinuity and the badly fluctuating part in the smooth region. This effect results from the multiplicative factor $\gamma_m(\theta_1(x))$ including the terms $|T_{1,h}(x-h)|$ and $|T_{1,h}(x+h)|$ in (9). In addition, log errors of $G^{[m]}_{1,h}(x)$ for the jump function $[f_1](x)$ are illustrated in Fig. 3, which implies that $G^{[m]}_{1,h}(x)$ converges faster away from the discontinuities as the order m of the sigmoidal transformation goes higher. Therein, log errors of $T_{1,h}(x)$ indicated by dotted lines are included for comparison with those of $G^{[m]}_{1,h}(x)$.

For a higher order edge detector, we consider the fifth order ($p = 2$) local edge detector as follows.

$$T_{5,h}(x)\frac{1}{6}\left\{-f_{j-2} + 5f_{j-1} - 10f_j + 10f_{j+1} - 5f_{j+2} + 5f_{j+3}\right\} \tag{16}$$

for all $x_j \le x < x_{j+1},\ j = 0, 1, \ldots, 2N$. The concentration property in (2) implies

$$T_{5,h}(x) = \begin{cases} [f](d) + O(h), & \text{if } d \in [x_j, x_{j+1}) \\ -\dfrac{2}{3}[f](d) + O(h), & \text{if } d \in [x_{j-1}, x_j) \cup [x_{j+1}, x_{j+2}) \\ \dfrac{1}{6}[f](d) + O(h), & \text{if } d \in [x_{j-2}, x_{j-1}) \cup [x_{j+2}, x_{j+3}) \\ O(h^5), & \text{otherwise} \end{cases} \tag{17}$$

as $h \to 0$. Results of higher order local edge detectors $T_{5,h}(x)$ and $G^{[m]}_{5,h}(x)$ are compared in Figs. 4 and 5. The results are consistent with the concentration property, (13) of $G^{[m]}_{5,h}(x)$. That is, the figures show that the presented edge detector $G^{[m]}_{5,h}(x)$, with m large enough, can eliminate sufficiently all the oscillatory points of $T_{5,h}(x)$ near each discontinuity as well as highly improve convergence to the jump function $[f_2](x)$ outside jump points.

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