Problem Sheet 1 (Topology): Solutions

Chapters 1–3, Hatcher: Basic Topology, Connectedness, and Compactness

I have used AI tools to help me edit this document.

I. Basic Point-Set Topology

1. **Closure and interior calculus.** Let A and B be subsets of a topological space X.

   (a) Prove that
   $$\overline{A \cup B} = \overline{A} \cup \overline{B}, \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

   (b) Prove that
   $$\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B), \qquad \operatorname{int}(A \cup B) \supseteq \operatorname{int}(A) \cup \operatorname{int}(B).$$

   (c) Give explicit examples in $\mathbb{R}$ for which the inclusions in the second formula of part (a) and the second formula of part (b) are strict.

Solution.

(a) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, monotonicity of closure gives
$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$
Conversely, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$,
$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$
Thus equality holds.

Also, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so
$$\overline{A \cap B} \subseteq \overline{A} \quad \text{and} \quad \overline{A \cap B} \subseteq \overline{B}.$$
Taking both inclusions together gives
$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

(b) The set $\operatorname{int}(A) \cap \operatorname{int}(B)$ is open and is contained in $A \cap B$. Hence
$$\operatorname{int}(A) \cap \operatorname{int}(B) \subseteq \operatorname{int}(A \cap B).$$
For the reverse inclusion, every open set contained in $A \cap B$ is contained in both A and B. In particular,
$$\operatorname{int}(A \cap B) \subseteq \operatorname{int}(A) \quad \text{and} \quad \operatorname{int}(A \cap B) \subseteq \operatorname{int}(B),$$
so
$$\operatorname{int}(A \cap B) \subseteq \operatorname{int}(A) \cap \operatorname{int}(B).$$
Finally, $\operatorname{int}(A)$ and $\operatorname{int}(B)$ are open subsets of $A \cup B$. Their union is therefore an open subset of $A \cup B$, and hence
$$\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B).$$

(c) For strictness in part (a), take

$$A = (0,1), \qquad B = (1,2).$$

Then $A \cap B = \varnothing$, so $\overline{A \cap B} = \varnothing$, while

$$\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}.$$

For strictness in part (b), take

$$A = [0,1], \qquad B = [1,2].$$

Then

$$\operatorname{int}(A) \cup \operatorname{int}(B) = (0,1) \cup (1,2),$$

whereas

$$\operatorname{int}(A \cup B) = \operatorname{int}([0,2]) = (0,2).$$

The point 1 belongs to the latter set but not to the former.

2. Open and closed maps. A map $f : X \to Y$ is called *open* if it sends open sets to open sets, and *closed* if it sends closed sets to closed sets.

(a) Give an example of an open map that is not closed, and an example of a closed map that is not open.

(b) Determine whether the coordinate projection

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad \pi(x,y) = x,$$

is open and whether it is closed.

(c) Determine whether

$$p : \mathbb{R} \longrightarrow S^1, \qquad p(t) = (\cos t, \sin t),$$

is open and whether it is closed.

Solution.

(a) The inclusion

$$i : (0,1) \hookrightarrow \mathbb{R}$$

is open: every open subset of $(0,1)$ has the form $V \cap (0,1)$ with V open in $\mathbb{R}$, and this intersection is open in $\mathbb{R}$. It is not closed, since $(0,1)$ is closed as a subset of itself, but its image $(0,1)$ is not closed in $\mathbb{R}$.

The inclusion

$$j : [0,1] \hookrightarrow \mathbb{R}$$

is closed. Indeed, if C is closed in $[0,1]$, then $C = F \cap [0,1]$ for some closed $F \subseteq \mathbb{R}$, so C is closed in $\mathbb{R}$. The map is not open, since the open subset $[0,1]$ of its own domain has image $[0,1]$, which is not open in $\mathbb{R}$.

(b) The projection π is open. If $U \subseteq \mathbb{R}^2$ is open and $x \in \pi(U)$, choose y such that $(x, y) \in U$. There is an $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon) \subseteq U.$$

It follows that $(x-\varepsilon, x+\varepsilon) \subseteq \pi(U)$. Thus every point of $\pi(U)$ is an interior point.

The projection is not closed. The set

$$C = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$$

is closed, since it is the inverse image of the closed set $\{1\}$ under the continuous map $(x, y) \mapsto xy$. However,

$$\pi(C) = \mathbb{R} \setminus \{0\},$$

which is not closed in $\mathbb{R}$.

(c) The map p is open. Let $U \subseteq \mathbb{R}$ be open and let $p(t) \in p(U)$, where $t \in U$. Choose $0 < \varepsilon < \pi$ such that

$$(t-\varepsilon, t+\varepsilon) \subseteq U.$$

The image of this interval is an open arc of S^1 containing $p(t)$. Hence every point of $p(U)$ has an open neighborhood in S^1 contained in $p(U)$.

The map is not closed. Let

$$C = \left\{2\pi n + \frac{1}{n} : n \in \mathbb{N}\right\}.$$

This set is closed in $\mathbb{R}$: it has no finite accumulation point, since its elements tend to $+\infty$. But

$$p(C) = \left\{\left(\cos\frac{1}{n}, \sin\frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

has $(1, 0)$ as a limit point, and $(1, 0) \notin p(C)$. Thus $p(C)$ is not closed in S^1.

3. The lower-limit topology. Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ equipped with the topology having the half-open intervals $[a, b)$ as a basis.

(a) For $A \subseteq \mathbb{R}_\ell$, prove that $x \in \overline{A}$ if and only if there is a sequence (x_n) in A such that

$$x_n \geq x \quad \text{for all } n, \qquad |x_n - x| \longrightarrow 0.$$

(b) Let the codomain $\mathbb{R}$ have its usual topology. Prove that a function $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous if and only if it is right-continuous at every point:

$$\lim_{\substack{t \to x \\ t > x}} f(t) = f(x).$$

Solution.

(a) Suppose first that $x \in \overline{A}$. For each $n \geq 1$, the basic neighborhood

$$\left[x, x + \frac{1}{n}\right)$$

meets A. Choose

$$x_n \in A \cap \left[x, x + \frac{1}{n}\right).$$

Then $x_n \geq x$ and $0 \leq x_n - x < 1/n$, so $|x_n - x| \to 0$.

Conversely, suppose such a sequence exists. If U is any neighborhood of x in $\mathbb{R}_\ell$, then for some $\varepsilon > 0$,

$$[x, x + \varepsilon) \subseteq U.$$

For all sufficiently large n, we have $x \leq x_n < x + \varepsilon$, so $x_n \in U \cap A$. Thus every neighborhood of x meets A, and $x \in \overline{A}$.

(b) Assume that f is continuous and fix $x \in \mathbb{R}$. Given $\varepsilon > 0$, continuity at x gives a neighborhood U of x in $\mathbb{R}_\ell$ such that

$$f(U) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon).$$

For some $\delta > 0$, $[x, x + \delta) \subseteq U$. Hence

$$x < t < x + \delta \quad \Longrightarrow \quad |f(t) - f(x)| < \varepsilon,$$

which is right-continuity at x.

Conversely, assume that f is right-continuous everywhere. Fix x and let V be an open neighborhood of $f(x)$ in the usual topology. Choose $\varepsilon > 0$ such that

$$(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq V.$$

Right-continuity gives $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ whenever $x < t < x + \delta$. This inequality also holds at $t = x$. Therefore

$$f([x, x + \delta)) \subseteq V.$$

Thus f is continuous at x as a map from $\mathbb{R}_\ell$, and hence is continuous everywhere.

4. **Quotient maps and identification spaces.** Let $q : X \to Y$ be a surjective map. The map q is called a *quotient map* if a subset $U \subseteq Y$ is open exactly when $q^{-1}(U)$ is open in X.

   (a) Suppose that q is continuous and open. Prove that q is a quotient map. Prove the same conclusion if q is continuous and closed.

   (b) Let $\sim$ be the equivalence relation on $[0, 1]$ that identifies 0 and 1 and leaves all other points distinct. Let

   $$q : [0, 1] \longrightarrow [0, 1]/\sim$$

   be the quotient map. Prove that the map

   $$f : [0, 1] \longrightarrow S^1, \qquad f(t) = (\cos 2\pi t, \sin 2\pi t),$$

   induces a well-defined continuous bijection

   $$\overline{f} : [0, 1]/\sim \longrightarrow S^1.$$

   (c) Prove that $\overline{f}$ is a homeomorphism. You may use the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

(d) Let Z be a topological space and let $g : [0,1] \to Z$ be continuous with $g(0) = g(1)$. Prove that there exists a unique continuous map

$$\overline{g} : [0,1]/\sim \longrightarrow Z$$

such that $g = \overline{g} \circ q$.

Solution.

(a) Assume first that q is continuous and open. If $U \subseteq Y$ is open, then $q^{-1}(U)$ is open by continuity. Conversely, if $q^{-1}(U)$ is open, then, using surjectivity,

$$U = q(q^{-1}(U)),$$

which is open because q is an open map. Thus q is a quotient map.

Now assume that q is continuous and closed. Again, U open implies $q^{-1}(U)$ open. Conversely, suppose $q^{-1}(U)$ is open. Then

$$q^{-1}(Y \setminus U) = X \setminus q^{-1}(U)$$

is closed. Since q is closed and surjective,

$$Y \setminus U = q(q^{-1}(Y \setminus U))$$

is closed. Hence U is open, and q is a quotient map.

(b) The map f is constant on every equivalence class: the only nontrivial class is $\{0,1\}$, and

$$f(0) = f(1) = (1,0).$$

Therefore

$$\overline{f}([t]) = f(t)$$

is well-defined, and $f = \overline{f} \circ q$.

To prove continuity, let $U \subseteq S^1$ be open. Then

$$q^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U),$$

which is open in $[0,1]$ because f is continuous. Since q is a quotient map, $\overline{f}^{-1}(U)$ is open. Thus $\overline{f}$ is continuous.

The map is surjective because every point of S^1 has the form $(\cos 2\pi t, \sin 2\pi t)$ for some $t \in [0,1]$. If $\overline{f}([s]) = \overline{f}([t])$, then

$$e^{2\pi i s} = e^{2\pi i t},$$

so $s - t \in \mathbb{Z}$. For $s, t \in [0,1]$, this means either $s = t$ or $\{s,t\} = \{0,1\}$. In either case $[s] = [t]$. Hence $\overline{f}$ is injective.

(c) The quotient map q is continuous, so $[0,1]/\sim = q([0,1])$ is compact as the continuous image of the compact interval $[0,1]$. The circle S^1 is Hausdorff because it is a subspace of $\mathbb{R}^2$. Therefore the continuous bijection $\overline{f}$ is a homeomorphism.

(d) Define

$$\overline{g}([t]) = g(t).$$

This is well-defined: the only distinct equivalent points are 0 and 1, and $g(0) = g(1)$. By construction, $g = \overline{g} \circ q$.

To prove continuity, let $U \subseteq Z$ be open. Then

$$q^{-1}(\overline{g}^{-1}(U)) = g^{-1}(U),$$

which is open in $[0, 1]$. Since q is a quotient map, $\overline{g}^{-1}(U)$ is open in $[0, 1]/\sim$. Thus $\overline{g}$ is continuous.

Finally, if $h : [0, 1]/\sim \to Z$ also satisfies $g = h \circ q$, then for every class $[t]$,

$$h([t]) = h(q(t)) = g(t) = \overline{g}([t]).$$

Thus $h = \overline{g}$, proving uniqueness.

II. Connectedness

5. Connectedness and closure. Let A be a connected subspace of a topological space X. Prove that its closure $\overline{A}$ is connected.

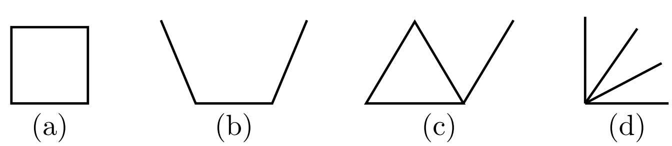
Solution. (We have proved this in class.) Suppose, toward a contradiction, that

$$\overline{A} = U \cup V$$

is a separation of $\overline{A}$. Thus U and V are disjoint, nonempty, and open in $\overline{A}$. The sets $A \cap U$ and $A \cap V$ are disjoint open subsets of A whose union is A. Since A is connected, one of them must be empty. After interchanging U and V if necessary, assume that $A \cap V = \varnothing$, so $A \subseteq U$.

But A is dense in $\overline{A}$, and V is a nonempty open subset of $\overline{A}$. Therefore V must meet A, contradicting $A \cap V = \varnothing$. Hence $\overline{A}$ is connected.

6. Cut points as homeomorphism invariants. A point x of a connected space X is a *cut point* if $X \setminus \{x\}$ is disconnected. Show that no two of the following four graphs are homeomorphic. Your proof should distinguish them by analyzing cut points, non-cut points, and branch behavior.



Solution. For a point in a finite graph, its *local branch number* is the number of components obtained by removing the point from a sufficiently small connected neighborhood containing no other vertex. An interior point of an edge has local branch number 2, an endpoint has local branch number 1, and a vertex at which k edges meet has local branch number k. Both the cut-point property and the local branch number are preserved by homeomorphisms.

In graph (a), removing any point leaves a connected arc. Thus graph (a) has no cut points.

Graph (b) is an arc. Its two endpoints are non-cut points, while every other point is a cut point. It has no branch point: every point has local branch number at most 2.

In graph (c), let v be the point where the extra edge is attached to the triangle. The point v, and every interior point of the extra edge, is a cut point. The free endpoint of the extra edge and all points of the triangle other than v are non-cut points. Moreover, v has local branch number 3.

In graph (d), the central vertex has local branch number 4. Every interior point of an arm is a cut point, while the four outer endpoints are non-cut points.

Consequently, (a) is distinguished by having no cut points; (b) has cut points but no branch point; and (c) and (d) are distinguished by the presence of a branch point of local branch number 3 versus 4. Hence no two of the four graphs are homeomorphic.

7. **Intermediate Value Theorem from connectedness.** Assume that every interval $[a, b] \subseteq \mathbb{R}$ is connected. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose

$$f(a) < c < f(b).$$

Prove that there exists $x \in [a, b]$ with $f(x) = c$. Your proof should use only connectedness and continuity, not completeness or a supremum argument.

Solution. The image $f([a, b])$ is connected because it is the continuous image of the connected set $[a, b]$. Suppose that no point of $[a, b]$ is mapped to c. Then

$$f([a, b]) \subseteq (-\infty, c) \cup (c, \infty).$$

The two sets

$$f([a, b]) \cap (-\infty, c) \quad \text{and} \quad f([a, b]) \cap (c, \infty)$$

are disjoint and open in the subspace $f([a, b])$. They are both nonempty because the first contains $f(a)$ and the second contains $f(b)$. They therefore form a separation of $f([a, b])$, contradicting connectedness. Hence $f(x) = c$ for some $x \in [a, b]$.

8. **Products of connected spaces.** Let X and Y be nonempty topological spaces.

(a) Fix $x_0 \in X$ and $y_0 \in Y$. Prove that

$$X \times \{y_0\} \cong X, \qquad \{x_0\} \times Y \cong Y.$$

(b) Assume that X and Y are connected. For each $y \in Y$, prove that

$$(X \times \{y\}) \cup (\{x_0\} \times Y)$$

is connected.

(c) Deduce that $X \times Y$ is connected whenever X and Y are connected.

(d) Suppose instead that X and Y are path-connected. Prove directly that $X \times Y$ is path-connected by constructing a path between two arbitrary points

$$(x_1, y_1), (x_2, y_2) \in X \times Y.$$

(e) Deduce that $\mathbb{R}^n$ is path-connected, and hence connected, for every $n \geq 1$.

Solution.

(a) The map

$$X \longrightarrow X \times \{y_0\}, \qquad x \longmapsto (x, y_0),$$

is a continuous bijection whose inverse is the restriction of the first coordinate projection. Hence it is a homeomorphism. The same argument, using the second coordinate projection, gives

$$Y \cong \{x_0\} \times Y.$$

(b) By part (a), the sets $X \times \{y\}$ and $\{x_0\} \times Y$ are connected. They intersect at the point (x_0, y). The union of two connected subsets with nonempty intersection is connected (proved in class), so

$$(X \times \{y\}) \cup (\{x_0\} \times Y)$$

is connected.

(c) For each $y \in Y$, set

$$C_y = (X \times \{y\}) \cup (\{x_0\} \times Y).$$

Each C_y is connected by part (b), and every C_y contains the common set $\{x_0\} \times Y$. Therefore their union is connected. Since

$$\bigcup_{y \in Y} C_y = X \times Y,$$

the product $X \times Y$ is connected.

(d) Choose paths

$$\alpha : [0,1] \to X, \qquad \alpha(0) = x_1, \quad \alpha(1) = x_2,$$

and

$$\beta : [0,1] \to Y, \qquad \beta(0) = y_1, \quad \beta(1) = y_2.$$

Then

$$\gamma : [0,1] \to X \times Y, \qquad \gamma(t) = (\alpha(t), \beta(t)),$$

is continuous and satisfies

$$\gamma(0) = (x_1, y_1), \qquad \gamma(1) = (x_2, y_2).$$

Thus $X \times Y$ is path-connected.

(e) Given $x, y \in \mathbb{R}^n$, the straight-line path

$$\gamma(t) = (1-t)x + ty, \qquad 0 \le t \le 1,$$

joins x to y. Hence $\mathbb{R}^n$ is path-connected. Every path-connected space is connected (proved in class), so $\mathbb{R}^n$ is connected as well.

III. Compactness

9. The finite-intersection characterization. Prove that a topological space X is compact if and only if the following property holds: whenever $\{F_\alpha\}_{\alpha\in I}$ is a family of closed subsets of X such that every finite subfamily has nonempty intersection, then

$$\bigcap_{\alpha\in I} F_\alpha \neq \varnothing.$$

Solution. Assume first that X is compact, and let $\{F_\alpha\}_{\alpha\in I}$ be a family of closed sets with the finite-intersection property. If

$$\bigcap_{\alpha\in I} F_\alpha = \varnothing,$$

then the open sets $X \setminus F_\alpha$ cover X. Compactness gives a finite subcover

$$X = (X \setminus F_{\alpha_1}) \cup \cdots \cup (X \setminus F_{\alpha_n}).$$

Taking complements yields

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \varnothing,$$

contradicting the finite-intersection property.

Conversely, assume the stated property and let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X. If it had no finite subcover, then for every finite set of indices $\alpha_1, \ldots, \alpha_n$,

$$X \setminus (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}) = (X \setminus U_{\alpha_1}) \cap \cdots \cap (X \setminus U_{\alpha_n})$$

would be nonempty. Thus the closed sets $F_\alpha = X \setminus U_\alpha$ would have the finite-intersection property. By assumption,

$$\bigcap_{\alpha\in I} F_\alpha \neq \varnothing.$$

But this intersection is the complement of $\bigcup_{\alpha\in I} U_\alpha = X$, a contradiction. Hence every open cover has a finite subcover, and X is compact.

10. Compactness as a homeomorphism obstruction.

(a) Construct a homeomorphism

$$[0,1) \times [0,1) \cong [0,1] \times [0,1).$$

You may first construct a homeomorphism between $[0,1)$ and a suitable broken line or ray-like subspace of the plane.

(b) Prove that $[0,1) \times [0,1)$ is not homeomorphic to $[0,1] \times [0,1]$.

Solution.

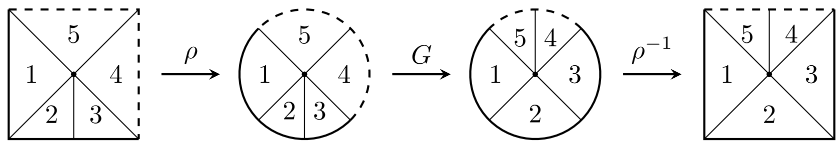
(a) We follow the sector-rearrangement construction that the students proposed in class! First use the affine homeomorphism

$$T : [0,1]^2 \longrightarrow [-1,1]^2, \qquad T(s,t) = (2s-1, 2t-1).$$

Thus it is enough to construct a homeomorphism

$$[-1,1)\times[-1,1)\longrightarrow[-1,1]\times[-1,1).$$

The deleted parts of the boundary are indicated by dashed lines in the following picture.



Let

$$Q=[-1,1]^2,\qquad D=\{x\in\mathbb{R}^2:\|x\|_2\le 1\}.$$

The radial map that rounds the square into the disk is

$$\rho:Q\longrightarrow D,\qquad \rho(0)=0,\qquad \rho(x)=\frac{\|x\|_\infty}{\|x\|_2}x\quad(x\neq 0),$$

where $\|x\|_2=\sqrt{x_1^2+x_2^2}$ is the Euclidean norm and $\|x\|_\infty=\max(x_1,x_2)$ is the Minkowski norm. This map preserves every ray from the origin and sends the boundary of the square to the unit circle. Its inverse is

$$\rho^{-1}(0)=0,\qquad \rho^{-1}(y)=\frac{\|y\|_2}{\|y\|_\infty}y\quad(y\neq 0).$$

Both maps are continuous at the origin because they preserve the radial parameter: $\|\rho(x)\|_2=\|x\|_\infty$ and $\|\rho^{-1}(y)\|_\infty=\|y\|_2$. Hence ρ is a homeomorphism.

We now rearrange the five sectors of the disk. Identify $\mathbb{R}^2$ with $\mathbb{C}$. Let φ be the strictly increasing, piecewise-linear function whose values at the sector rays are

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$
$\varphi(\theta)$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$.

Extend it to all real numbers by

$$\varphi(\theta+2\pi)=\varphi(\theta)+2\pi.$$

Define

$$G(0)=0,\qquad G\big(re^{i\theta}\big)=re^{i\varphi(\theta)}\quad(0<r\le 1).$$

This is well defined, and it is a homeomorphism of D; its inverse is obtained by replacing φ with φ^{-1}. It sends each numbered sector in the first disk homeomorphically onto the sector with the same number in the second disk.

Under ρ, the points of the boundary belonging to $[-1,1)\times[-1,1)$ form the open circular arc

$$\left\{e^{i\theta}:\frac{3\pi}{4}<\theta<\frac{7\pi}{4}\right\}.$$

The map G sends this arc to

$$\left\{e^{i\theta}:\frac{3\pi}{4}<\theta<\frac{9\pi}{4}\right\},$$

which is exactly the part of the boundary belonging to $[-1,1] \times [-1,1)$. Thus

$$\rho^{-1} \circ G \circ \rho : [-1,1) \times [-1,1) \longrightarrow [-1,1] \times [-1,1)$$

is a homeomorphism. Returning to the original coordinates, the required map is

$$T^{-1} \circ \rho^{-1} \circ G \circ \rho \circ T.$$

(b) The square $[0,1] \times [0,1]$ is compact. We show that $[0,1) \times [0,1)$ is not compact. The first-coordinate projection

$$\pi_1 : [0,1) \times [0,1) \longrightarrow [0,1)$$

is continuous and surjective. If the product were compact, then its continuous image $[0,1)$ would be compact. But $[0,1)$ is not compact: the open sets

$$U_n = \left[0, 1 - \frac{1}{n}\right), \qquad n \geq 2,$$

are open in the subspace $[0,1)$, cover $[0,1)$, and admit no finite subcover. Hence $[0,1) \times [0,1)$ is noncompact. Since compactness is preserved by homeomorphisms, it cannot be homeomorphic to $[0,1] \times [0,1]$.

(b) The square $[0,1] \times [0,1]$ is compact. We show that $[0,1) \times [0,1)$ is not compact. The first-coordinate projection

$$\pi_1 : [0,1) \times [0,1) \longrightarrow [0,1)$$

is continuous and surjective. If the product were compact, then its continuous image $[0,1)$ would be compact. But $[0,1)$ is not compact: the open sets

$$U_n = \left[0, 1 - \frac{1}{n}\right), \qquad n \geq 2,$$

are open in the subspace $[0,1)$, cover $[0,1)$, and admit no finite subcover. Hence $[0,1) \times [0,1)$ is noncompact. Since compactness is preserved by homeomorphisms, it cannot be homeomorphic to $[0,1] \times [0,1]$.

11. Distance between compact sets. Let (X,d) be a metric space, and for nonempty subsets $A, B \subseteq X$ define

$$d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}.$$

(a) If A and B are compact, prove that there exist $a_0 \in A$ and $b_0 \in B$ such that

$$d(A,B) = d(a_0,b_0).$$

(b) Deduce that if A and B are disjoint compact sets, then $d(A,B) > 0$.

(c) Give examples showing that both conclusions can fail when A and B are merely closed rather than compact.

Solution.

(a) The product $A \times B$ is compact. The function

$$D : A \times B \longrightarrow \mathbb{R}, \qquad D(a,b) = d(a,b),$$

is continuous, since the triangle inequality gives

$$|d(a,b) - d(a',b')| \leq d(a,a') + d(b,b').$$

A continuous real-valued function on a compact metric space attains its minimum. Hence there is $(a_0, b_0) \in A \times B$ such that

$$d(a_0, b_0) = \min_{(a,b) \in A \times B} d(a,b) = d(A,B).$$

(b) By part (a), $d(A,B) = d(a_0, b_0)$ for some $a_0 \in A$ and $b_0 \in B$. If $d(A,B) = 0$, then $d(a_0, b_0) = 0$, so $a_0 = b_0$. This would give a point in $A \cap B$, contrary to disjointness. Therefore $d(A,B) > 0$.

(c) In $\mathbb{R}^2$ with the Euclidean metric, let

$$A = \{(x,0) : x \geq 0\}, \qquad B = \left\{ \left(x, \frac{1}{x+1} \right) : x \geq 0 \right\}.$$

Both sets are closed. They are disjoint, but

$$d\left((x,0), \left(x, \frac{1}{x+1} \right) \right) = \frac{1}{x+1} \longrightarrow 0.$$

Thus $d(A,B) = 0$, showing that disjoint closed sets need not have positive distance. Since no pair of points in the disjoint sets can have distance 0, the infimum is not attained. The same example therefore shows that the conclusion of part (a) can also fail for closed noncompact sets.

12. The diagonal and the Hausdorff property. Let X be a topological space, and let

$$\Delta_X = \{(x,x) : x \in X\} \subseteq X \times X$$

be the diagonal.

(a) Prove that if X is Hausdorff, then Δ_X is closed in $X \times X$.

(b) Prove the converse: if Δ_X is closed in $X \times X$, then X is Hausdorff.

(c) Let Z be a topological space, let X be Hausdorff, and let

$$f, g : Z \to X$$

be continuous. Prove that the equalizer

$$\{z \in Z : f(z) = g(z)\}$$

is closed in Z.

Solution.

(a) Let $(x, y) \notin \Delta_X$, so $x \neq y$. Since X is Hausdorff, there are disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$. Then $U \times V$ is an open neighborhood of (x, y) in $X \times X$. It does not meet the diagonal: if $(z, z) \in U \times V$, then $z \in U \cap V$, which is impossible. Thus every point outside Δ_X has an open neighborhood disjoint from Δ_X, so $(X \times X) \setminus \Delta_X$ is open and Δ_X is closed.

(b) Assume that Δ_X is closed, and let $x \neq y$. Then (x, y) belongs to the open set $(X \times X) \setminus \Delta_X$. By the definition of the product topology, there are open sets $U, V \subseteq X$ such that

$$x \in U, \qquad y \in V, \qquad U \times V \subseteq (X \times X) \setminus \Delta_X.$$

The sets U and V must be disjoint. Otherwise, if $z \in U \cap V$, then $(z, z) \in U \times V$, contradicting the last inclusion. Thus distinct points of X have disjoint neighborhoods, and X is Hausdorff.

(c) Define

$$F : Z \longrightarrow X \times X, \qquad F(z) = (f(z), g(z)).$$

This map is continuous. Moreover,

$$\{z \in Z : f(z) = g(z)\} = F^{-1}(\Delta_X).$$

Since X is Hausdorff, part (a) shows that Δ_X is closed in $X \times X$. Therefore its inverse image under the continuous map F is closed in Z.