Problem Sheet 2 (Do Carmo: Chapter 3, Chapter 1)

Differentiable Manifolds, calculus of differential forms

I. Examples of Differentiable Manifolds

1. **Real projective space.** Define real projective n-space by

$$\mathbb{RP}^n = \big(\mathbb{R}^{n+1} \setminus \{0\}\big)/\sim,$$

where

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \qquad (\lambda \in \mathbb{R} \setminus \{0\}).$$

Write $[x_0 : \cdots : x_n]$ for the equivalence class of $(x_0, \dots, x_n)$. For $i = 0, \dots, n$, let

$$V_i = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}.$$

Define a map $\varphi_i : V_i \to \mathbb{R}^n$ by dividing all coordinates by x_i and omitting the ith coordinate.

(a) Prove that φ_i is well defined and bijective, and write down φ_i^{-1} explicitly.

(b) Prove that the sets $V_0, \dots, V_n$ cover $\mathbb{RP}^n$.

(c) For $i \neq j$, determine the open set $\varphi_i(V_i \cap V_j) \subseteq \mathbb{R}^n$ and compute the transition map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \longrightarrow \varphi_j(V_i \cap V_j).$$

Prove that this transition map is smooth.

(d) Conclude that the charts (V_i, φ_i) give $\mathbb{RP}^n$ the structure of a differentiable n-manifold.

(e) Show that $\mathbb{RP}^1$ is diffeomorphic to the circle S^1.

2. **The product of two differentiable manifolds.** Let M^m and N^n be differentiable manifolds. Suppose that

$$\{(U_\alpha, \varphi_\alpha)\} \qquad \text{and} \qquad \{(V_\beta, \psi_\beta)\}$$

are differentiable atlases for M and N, respectively. For each pair (α, β), define

$$\Phi_{\alpha\beta} : U_\alpha \times V_\beta \longrightarrow \mathbb{R}^{m+n}$$

by

$$\Phi_{\alpha\beta}(p, q) = \big(\varphi_\alpha(p), \psi_\beta(q)\big).$$

(a) Prove that the sets $U_\alpha \times V_\beta$ cover $M \times N$ and that each $\Phi_{\alpha\beta}$ is a homeomorphism onto an open subset of $\mathbb{R}^{m+n}$.

(b) Compute the transition map between two overlapping product charts and prove that it is smooth.

(c) Deduce that $M \times N$ has a natural structure of a differentiable manifold of dimension $m + n$.

(d) Prove that the coordinate projections

$$\pi_M : M \times N \to M, \qquad \pi_N : M \times N \to N$$

are smooth.