MATH401 Midterm 1

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- Put away everything except a pen/pencil, eraser.
- This exam contains 7 problems.
- Argue each step of your solution.
- Use the back of the pages as scrap paper.
- Good luck!

Problem 1 (10 points). *Find the LU factorization of the following matrix:*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -3 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -3 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & -7 \end{bmatrix}}_{=:U}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & -7 \end{bmatrix}$$

Problem 2. *(i)[10 points] Determine the inverse of the following matrix*

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

(ii)[5 points] Possibly using (i), solve the system

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

(i)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = -R_3]{R_2 = -R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[R_1 = R_1 - 2R_3]{R_2 = R_2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 = \frac{R_1}{2}}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & & & -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ & 1 & & 2 & -1 & -1 \\ & & 1 & 2 & -1 & 0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

(ii)

$$A x = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \to \underbrace{(A^{-1} A) x}_{= x} = A^{-1} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

Problem 3. *(i)[10 points] Find the basis and dimension of the corange and cokernel of the following matrix*

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 4 \\ -3 & 9 & 1 \end{bmatrix}$$

(ii)[5 points] Based on part (i), what is the dimension of the kernel and range of A?

$$A^t = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 0 & 4 & 1 \end{bmatrix}$$

(i) $\operatorname{coker} A = \ker A^t = \{ x \in \mathbb{R}^3 \text{ s.t. } A^t x = 0 \}$

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + 3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \boxed{1} & 2 & -3 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free

$x_1 + 2x_2 - 3x_3 = 0$

$4x_2 + x_3 = 0$

$\Rightarrow \boxed{x_2 = \frac{-x_3}{4}} \Rightarrow$

$\Rightarrow x_1 = 3x_3 + \frac{2}{4} x_3$

$\boxed{x_1 = \frac{7}{2} x_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} x_3 \\ -\frac{1}{4} x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

basis of coker A

$\boxed{\dim \operatorname{coker} A = 1}$

$\operatorname{corng} A = \operatorname{rng} A^t = \operatorname{Col} A^t$

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 9 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{(as before)}} \begin{bmatrix} \boxed{1} & 2 & -3 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{base of corng A is } \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}; \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix} \Rightarrow$$

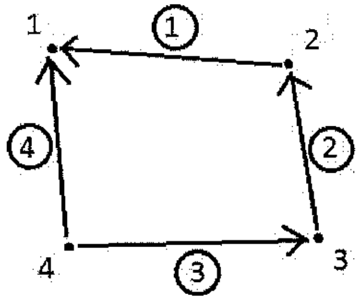
$\Rightarrow \boxed{\dim \operatorname{corng} A = 2}$

(ii) $\dim \operatorname{rng} A = \dim \operatorname{corng} A \overset{\text{part(i)}}{=} 2$

[Fundamental theorem of lin algebra]

$\dim \operatorname{Ker} A = 3 - \dim \operatorname{rng} A = 1$

Problem 4. *You are given the following digraph:*



(i)[5 points] Write down the incidence matrix A of the digraph.
(ii)[5 points] Argue that the cokernel of A has dimension 1 and find its basis vector v.
(iii)[5 points] Explain what information the vector v contains about the graph G.

(i)
$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \to A^t = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(ii)
$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 = R_1 + R_2]{R_1 := -R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_2 + R_3]{R_2 := -R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & +1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & +1 & 1 \end{bmatrix} \to$$

$$\xrightarrow[R_4 = R_4 + R_3]{R_3 := -R_3} \begin{bmatrix} \boxed{1} & 0 & 0 & +1 \\ 0 & \boxed{1} & 0 & +1 \\ 0 & 0 & \boxed{1} & +1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{v}$$

$\uparrow$ free

(iii) v represents the only circuit of the digraph:
→ we need to walk in the indicated direction on edge (4)
→ we need to walk in the opposing direction on the edges (1) (2) (3)

Problem 5. *In the spring mass system below the masses m_1 and m_2 are pulling the springs by a force of 1 Newton and 2 Newtons respectively. The spring constants are equal to 1.*

(i)[10 points]Using linear algebra, compute the displacement of the masses.
(ii)[5 points] Using linear algebra, compute the elongation of each spring.

(i)

$$e_1 = u_1$$
$$e_2 = u_2 - u_1$$
$$e_3 = -u_2$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} +1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

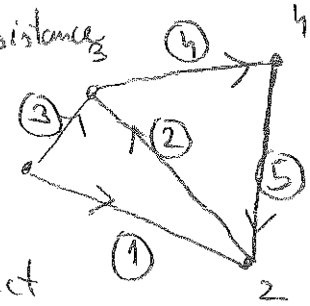
$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 2 \end{array}\right] \xrightarrow[R_1 = -R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & -2 \\ 2 & -1 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 - 2R_1]{} \left[\begin{array}{cc|c} 1 & -2 & -2 \\ 0 & 3 & 5 \end{array}\right] \Rightarrow \begin{array}{l} u_1 - 2u_2 = -2 \\ 3u_2 = 5 \end{array}$$

$$\Rightarrow \boxed{u_2 = \frac{5}{3}} \Rightarrow \boxed{u_1 = \frac{10}{3} - 2 = \frac{4}{3}}$$

displacement of masses $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \end{bmatrix}$

$\boxed{e = Au}$ (ii) $\boxed{e_1 = \frac{4}{3}}$ $\boxed{e_2 = \frac{5}{3} - \frac{4}{3} = \frac{1}{3}}$ $\boxed{e_3 = -\frac{5}{3}}$

Problem 6: each wire has $\frac{1}{2}$ Ohm resistances



(i)

The resistivity matrix is a product

$$K = A^T C \cdot A$$

$$A = \begin{bmatrix} +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & -1 & 0 & +1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

incidence matrix

$$K = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix}$$

(ii) Find general solution of the system $f = K \cdot u$:

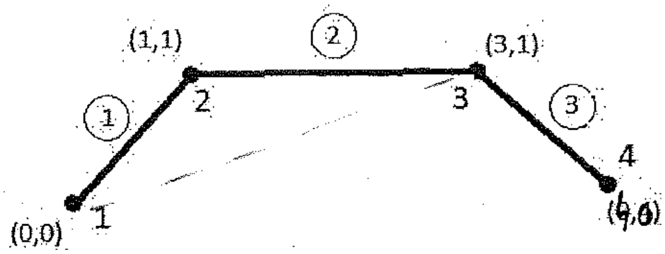
$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

There will be one free variable.
Find u_2, u_3, u_4 so that $u_1 = 0$.

Other method: as $u_1 = 0$, we can delete the first column of K and solve the system:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} \rightarrow \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}$$

Problem 7. *You are given the following structure, with the coordinates of the joints included:*



(i) *[5 points] Write down the incidence matrix A of the structure*

(ii) *[5 points] Find the reduced incidence matrix A^* if the bottom joints are grounded.*

(iii) *[5 points] Explain in words, what needs to be checked to see if the structure of part (ii) is stable. Finally, write down the reinforced incidence matrix A^{**} if you add in a rod connecting the first and third joints.*

(i)
$$A = \begin{array}{cccccccc} \multicolumn{2}{c}{u_1} & \multicolumn{2}{c}{u_2} & \multicolumn{2}{c}{u_3} & \multicolumn{2}{c}{u_4} \end{array}$$
$$A = \left[\begin{array}{cc|cc|cc|cc} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right]$$

$$e_1 = n_1 \cdot (u_1 - u_2) = \frac{(-1,-1)}{\sqrt{2}}(u_1 - u_2)$$
$$e_2 = n_2 \cdot (u_2 - u_3) = \frac{(-2,0)}{2}(u_2 - u_3)$$
$$e_3 = n_3 \cdot (u_3 - u_4) = \frac{(-1,1)}{\sqrt{2}}(u_3 - u_4)$$

(ii) Erase the first two and last two columns:
$$A^* = \left[\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]$$

(iii) The structure is stable if $A^* x = 0$ has only the trivial solution $\longleftrightarrow$ $(\operatorname{Ker} A^* = \{0\})$

$$A^{**} = \left[\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{array}\right]$$

$$e_4 = n_4 \cdot (u_1 - u_2) = \frac{(-3,-1)}{\sqrt{10}}(u_2 - u_3)$$