MATH461 Midterm 1

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Name: TAMAS DARVAS

- Put away everything except a pen/pencil, eraser.
- This exam contains 7 problems.
- Argue each step of your solution.
- Use the back of the pages as scrap paper.
- Good luck!

Problem 1 (10 points). *Indicate whether the following statements are true of false:*

(i) A homogeneous linear system is always consistent. T

(ii) A consistent linear system composed of four equations and five variables has infinitely many solutions. T

(iii) A linear system has a unique solution if there are as many pivots as there are equations. F

(iv) A linear map $F : \mathbb{R}^m \to \mathbb{R}^n$ is onto if the columns of its matrix are linearly dependent. F

(v) A linear map $F : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one if the columns of its matrix span $\mathbb{R}^n$. F

Problem 2 (15 points). *Determine if the following system is consistent. If consistent, determine the general solution and write it into parametric vector form.*

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -4x_1 - 4x_2 - 8x_3 + 4x_4 = -16 \\ -3x_2 - 3x_3 + 3x_4 = 12. \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 8 \\ -4 & -4 & -8 & 4 & -16 \\ 0 & -3 & -3 & 3 & 12 \end{array}\right] \begin{array}{l} R_1 = \frac{R_1}{2} \\ R_2 = \frac{R_2}{-4} \\ \sim \\ R_3 = \frac{R_3}{-3} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 4 \\ 1 & 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -1 & -4 \end{array}\right] \overset{R_2 = R_2 - R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & -4 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ \sim \\ R_3 = -R_3 \end{array} \begin{bmatrix} 1 & 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & +1 & 0 \end{bmatrix} \overset{R_1 = R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 & 8 \\ 0 & 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{R_2 = R_2 + R_3}{R_1 = R_1 - R_3}} \begin{bmatrix} 1 & & 1 & 0 & 8 \\ & 1 & 1 & 0 & -4 \\ & & & 1 & 0 \end{bmatrix}$$

consistent

x_3 free variable

solution: $\begin{cases} x_1 = 8 - x_3 \\ x_2 = -4 - x_3 \\ x_3 \text{ free} \\ x_4 = 0 \end{cases}$

$\downarrow$

parametric vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Problem 3 (15 points). *For what values of h is the following system inconsistent, consistent with unique solution or consistent with infinitely many solutions?*

$$\begin{cases} 2x_1 + 2x_3 = h \\ -x_2 + x_3 = 2 + h \\ x_1 + x_2 = 3. \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & h \\ 0 & -1 & 1 & 2+h \\ 1 & 1 & 0 & 3 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & 2+h \\ 2 & 0 & 2 & h \end{array}\right] \underset{R_2 = -R_2}{\overset{R_3 = R_3 - 2R_1}{\sim}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & -2-h \\ 0 & -2 & 2 & h-6 \end{array}\right] \underset{R_3 = R_3 + 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & -2-h \\ 0 & 0 & 0 & \underbrace{h-6-4-2h}_{= -10-h} \end{array}\right]$$

echelon form

Consistent if $-10 - h = 0 \iff \boxed{h = -10}$

- There are only two pivots so solution is <u>never</u> unique.
- If $h = -10$, then system is consistent with infinitely many solutions.
- If $h \neq -10$, then system is inconsistent.

Problem 4. *You are given the following vectors*

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

(i) (5 points) Is $\{v_1, v_2, v_3\}$ linearly independent? Argue your answer.

(ii) (5 points) Is the vector $b = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ in the span of $\{v_1, v_2, v_3\}$? If yes, write b as a linear combination of v_1, v_2, v_3.

(iii) (5 points) Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$? Argue your answer.

(i) $\{v_1, v_2, v_3\}$ is linearly independent if the following vector equation only has the trivial solution

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Aug. matrix: $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\{v_1, v_2, v_3\}$ linearly independent $\leftarrow$ 3 pivots, hence solution is trivial

(ii) To answer the question, we have to find a solution to the following vector equation: $x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Aug matrix: $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -1 & 2 & -1 & 2 \\ 1 & 1 & 0 & -2 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[R_1 = R_1 + R_3]{R_2 = R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right]$

$\hookrightarrow$ consistent

$$\xrightarrow{R_2 = \frac{R_2}{3}} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow[R_1 = R_1 - R_2]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{cases} x_3 = -3 \\ x_2 = -1 \\ x_1 = -1 \end{cases}$$

Hence $b = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

(iii) $\{v_1, v_2, v_3\}$ <u>does</u> span $\mathbb{R}^3$, because for any $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the system

$$\begin{cases} x_1 + x_2 - x_3 = b_1 \\ -x_1 + 2x_2 - x_3 = b_2 \\ x_1 + x_2 = b_3 \end{cases}$$

$\rightarrow$ has 3 pivots as seen in part (i) and (ii).

$\downarrow$

hence it always has a solution.

Problem 5 (15 points). *You are given the following vectors*

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \; v_3 = \begin{bmatrix} -1 \\ -1+2h \\ h \end{bmatrix}$$

For what values h is the set $\{v_1, v_2, v_3\}$ linearly independent?

v_1, v_2, v_3 are independent if the vector equation:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1+2h \\ h \end{bmatrix} = 0$$

has only the trivial solution.

$$\left[\begin{array}{cccc} 1 & 1 & -1 & 0 \\ -1 & 2 & -1+2h & 0 \\ 1 & -2 & h & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \left[\begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & 3 & -2+2h & 0 \\ 0 & -3 & h+1 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 + R_2}$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & 3 & -2+2h & 0 \\ 0 & 0 & -1+3h & 0 \end{array}\right] \rightarrow$$ You have only the trivial solution if you have exactly 3 pivots.

$\Updownarrow$

$-1+3h \neq 0 \iff h \neq \frac{1}{3}$

If $h \neq \frac{1}{3}$ then $\{v_1, v_2, v_3\}$ is linearly independent.

Problem 6. *(i) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation that satisfies*

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Write down the matrix of the transformation. Compute $T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$.

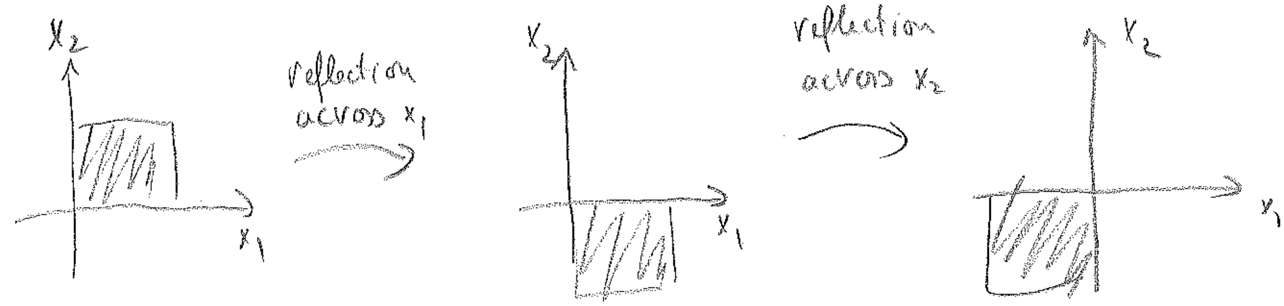
(ii) (5 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by first reflecting across the x_1–axis and then reflecting across the x_2–axis. What is the matrix of L?

(i) The matrix of the Transformation T is

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$$

(ii)



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The matrix is $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Problem 7. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation $Tx = Ax$, with matrix*

$$A = \begin{bmatrix} 1 & -5 & -3 \\ -1 & 4 & 2 \\ 2 & -5 & -1 \end{bmatrix}$$

(i) (5 points) Let $b = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Is b in the range of T? In other words, does there exist $x \in \mathbb{R}^3$ such that $Tx = b$? If x exists, is it unique?

(ii) (5 points) Is T a one-to-one map? Argue your answer.

(iii) (5 points) Does T map $\mathbb{R}^3$ onto $\mathbb{R}^3$? Argue your answer.

(i) Need to solve the matrix equation $Ax = b$

$$\begin{bmatrix} 1 & -5 & -3 \\ -1 & 4 & 2 \\ 2 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \text{Aug matrix: } \begin{bmatrix} 1 & -5 & -3 & -2 \\ -1 & 4 & 2 & 1 \\ 2 & -5 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + R_1}$$

$$\begin{bmatrix} 1 & -5 & -3 & -2 \\ 0 & -1 & -1 & -1 \\ 0 & 5 & 5 & 5 \end{bmatrix} \xrightarrow[R_2 = -R_2]{R_3 = R_3 - 5R_2} \begin{bmatrix} 1 & -5 & -3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$

Consistent with infinitely many solutions.

b is in the range of T!

the solution to $Tx = b$ is not unique!

(ii) As there are many solutions to $Tx=b$, T is <u>not one-to-one</u>

<u>Alternative explanation</u>: the matrix A has less pivots than columns.

(iii) The matrix A has less pivots than rows, hence it is <u>not onto</u>.

<u>Alternative explanation</u>: The columns of A do not span $\mathbb{R}^3$.