MATH461 Midterm 2

April 2.

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- Put away everything except a pen/pencil, eraser.
- This exam contains 8 problems.
- Argue each step of your solution.
- Use the back of the pages as scrap paper.
- Good luck!

Problem 1 (14 points). *Indicate whether the following statements are true of false:*

(i) *The first octant is a subspace of* $\mathbb{R}^3$. False

(ii) *Every plane is a subspace of* $\mathbb{R}^3$. False

(iii) *The* Ox *axis is a subspace of* $\mathbb{R}^2$. True

(iv) *A linearly independent set in a subspace* H *is a basis of* H. False

(v) *Given a matrix* A*, the columns of* $rrefA$ *without pivots form a basis of* $NulA$. False

(vi) *If* A *is a* 5×4 *matrix then* $NulA$ *contains a non-zero vector.* False

(vii) *After making two row exchanges the determinant of a matrix stays the same.* True

Problem 2. *You are given the following matrix:*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 2 & -1 & -1 \end{bmatrix}$$

(i) *(3 points) Compute* $\det A$.

(ii) *(6 points) Compute* A^{-1}.

(iii) *(3 points) Verify that* $A^{-1}A = I_3$*, where* I_3 *is the identity matrix.*

(i)

$$\det(A) = \begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 2 \\ 2 & -1 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} = -1 - 2 + 4 = \boxed{1}$$

(ii)

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A) = \operatorname{adj} A = \boxed{\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix}}$$

(iii)

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 1+(-1) & 1+(-1) \\ 2-2 & -2+3 & -2+2 \\ -4+2+2 & 4-3-1 & 4-2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3. *(i) (6 points) You are given the following points in* $\mathbb{R}^2$:

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Compute the area of the triangle ABC_Δ.

(ii) (6 points) You are given the following points in $\mathbb{R}^3$:

$$A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

Compute the volume of the parallelopiped determined by these 3 points and the origin.

(i) Translate points, so that A becomes the origin then

$$B' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$C' = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\text{Area } ABC_\Delta = \frac{1}{2}\left|\det\begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}\right| = \frac{6}{2} = \underline{\underline{3}}$$

(ii) $$\text{Volume} = \left| \det \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} \right| = \left| 1 \cdot 1 - 1 \cdot 1 - 1(-3) \right| = 3$$

Problem 4. *Let $H = span\{v_1, v_2, v_3, v_4, v_5\}$, where*

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; v_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_5 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

(i) (8 points) Find a basis of H.

(ii) (4 points) Is it true that $H = \mathbb{R}^3$? Argue your answer!

(i) $$\begin{bmatrix} 2 & 1 & 0 & 1 & 4 \\ 1 & 0 & 1 & 1 & 5 \\ -1 & -1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 5 \\ 2 & 1 & 0 & 1 & 4 \\ -1 & -1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & -2 & -1 & -6 \\ 0 & -1 & 2 & 1 & 6 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & -2 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a base of H

(ii) $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ spans a plane in $\mathbb{R}^3$, hence H is a plane in $\mathbb{R}^3$, $H \neq \mathbb{R}^3$.

Problem 5. *We define $F : \mathbb{P} \to \mathbb{P}_3$ by the general formula*

$$F(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_p t^p) = a_0 + a_1 t^1 + a_3 t^3.$$

For example: $F(1 + t^2 - t^3 + t^4) = 1 - t^3$.

(i) *(7 points) Show that F is a linear transformation.*

(ii) *(5 points) What polynomials are contained in the kernel of F? Argue your answer!*

(i) We need to show that

$F(p_1 + p_2) = F(p_1) + F(p_2)$ and $F(c \cdot p_1) = c \cdot F(p_1)$

where p_1, p_2 are arbitrary polynomials and c is any scalar.

Let $p_1 = a_0 + a_1 t + \ldots + a_p t^p$ and $p_2 = b_0 + b_1 t + \ldots + a_\ell \cdot t^\ell$

Then $\left.\begin{aligned} F(p_1) &= a_0 + a_1 t + a_3 t^3 \\ F(p_2) &= b_0 + b_1 t + b_3 t^3 \end{aligned}\right\} (*)$

Also $F(p_1 + p_2) = F\left((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3 + \ldots\right) =$

$= (a_0 + b_0) + (a_1 + b_1)t + (a_3 + b_3)t^3 \quad (**)$

From $(*)$ and $(**)$ it follows that $\boxed{F(p_1) + F(p_2) = F(p_1 + p_2)}$

$\left.\begin{aligned} c \cdot F(p_1) &= c(a_0 + a_1 t + a_3 t^3) = ca_0 + ca_1 t + ca_3 t^3 \\ F(cp_1) &= F(ca_0 + ca_1 t + \ldots + ca_p t^p) = ca_0 + ca_1 t + ca_3 t^3 \end{aligned}\right\} \Rightarrow \boxed{F(cp_1) = cF(p_1)}$

ii) $p \in \operatorname{Ker} F$ if $F(p) = \boxed{a_0 + a_1 t + a_3 t^3 = 0}$

$p = a_0 + a_1 t + \ldots + a_p t^p$

$\boxed{\text{Hence } p \in \operatorname{Ker} F \text{ if the coefficients } a_0, a_1, a_3 \text{ are } \underline{\text{zero}}.}$

Problem 6. *(i) (7 points) H is the collection of vectors that are given by the following experssion*

$$\begin{bmatrix} a+b+c \\ -2a+b \\ 2a-c \end{bmatrix},$$

where a, b, c are real numbers. Why is H a subspace of $\mathbb{R}^3$? Find a basis of H!

(ii) (7 points) K is the collection of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *such that $x_1 - x_2 + x_3 = 0$ and $2x_2 + 4x_3 = 0$.*

Why is K a subspace of $\mathbb{R}^3$? Find a basis of K!

(i) $\begin{bmatrix} a+b+c \\ -2a+b \\ 2a+c \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad a, b, c \in \mathbb{R}$

Hence $H = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

The span of a collection of vectors is a subspace of $\mathbb{R}^3$; hence H is a subspace of $\mathbb{R}^3$.

<u>Find a basis of H:</u>

$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

is a base of H.

(ii) $\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_2 + 4x_3 = 0 \end{cases} \longleftrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow K = \text{Nul } A$ where $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$

<u>$K = \text{Nul } A$ is a subspace of $\mathbb{R}^3$</u> !

<u>Basis of Nul A:</u>

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow[R_2 = \frac{1}{2} R_2]{R_1 = R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \to \boxed{\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a base of Nul } A = K}$

$\hookrightarrow x_3$ free

Problem 7. *You are given the following matrix:*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

(i) (6 points) Find a basis of NulA.

(ii) (6 points) Is the vector $v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ *contained in ColA? Argue your answer!*

(i) Need to find solutions of $Ax = 0$!

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2]{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$\hookrightarrow x_3$ free

base of Nul A is $\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}$

(ii) To answer the question, we need to see if the following equation has a solution:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & -2 & 4 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - \frac{R_2}{2}} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \Rightarrow \text{Inconsistent!}$$

$\downarrow$

$v \notin \text{Col}A$!

Problem 8. *(12 points) For this problem you are dealing with $\mathbb{P}_2$, the vectorspace of polynomials of degree at most 2. Let p_1, p_2, p_3 be the following polynomials:*

$$p_1 = 1 + 2t + t^2,\ p_2 = 1 - t,\ p_3 = 1 + t^2.$$

Is $\mathcal{B} = \{p_1, p_2, p_3\}$ a basis of $\mathbb{P}_2$? Argue your answer.

Solution: B is a basis if it is linearly independent + spans.

Linear independency: Suppose

$$x_1 \cdot p_1 + x_2 \cdot p_2 + x_3 p_3 = 0$$

$$\Leftrightarrow x_1(1+2t+t^2) + x_2(1-t) + x_3(1+t^2) = 0$$

$$\Leftrightarrow (x_1+x_2+x_3) + (2x_1 - x_2)t + (x_1+x_3)t^2 = 0$$

$$\Leftrightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R_2 = -\frac{1}{2}R_2]{R_2 \leftrightarrow R_3}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$\hookrightarrow x_1 = x_2 = x_3 = 0$

$\Downarrow$

p_1, p_2, p_3 is linearly independent.

Spanning: given $p = a_0 + a_1 t + a_2 t^2$, we need to show that there exists $x_1, x_2, x_3 \in \mathbb{R}$ s.t. $\boxed{x_1 p_1 + x_2 p_2 + x_3 p_3 = p}$

$\Downarrow$

As above, this last equation is equivalent to a linear system with 3 equations and 3 pivots, hence it always has a solution, hence $\{p_1, p_2, p_3\}$ spans $\mathbb{P}_2$!