# MATH461 Midterm 3

May 7.

Name: TAMAS DARVAS

- Put away everything except a pen/pencil, eraser.
- This exam contains 8 problems.
- Argue each step of your solution.
- Use the back of the pages as scrap paper.
- Good luck!

**Problem 1** (16 points). *Indicate whether the following statements are true of false:*

(i) *If a subspace $W$ is spanned by 3 vectors then* $\dim W = 3$. F

(ii) *An $n \times n$ matrix $A$ is always diagonalizable.* F

(iii) $\mathbb{R}^5$ *does not have 3 dimensional subspaces.* F

(iv) *For any matrix $A$ we have* $\dim ColA^T = \dim RowA$. T

(v) $rankA + dimNulA = n$, *for any $m \times n$ matrix $A$.* T

(vi) *A vector $v$ is in the orthogonal complement of a subspace $W = span\{u_1, u_2, \ldots, u_n\}$ if $v$ is orthogonal to each of the vectors $u_1, u_2, \ldots, u_n$.* T

(vii) *If the orthogonal projection of a vector $y$ onto a subspace $W$ is $y$ itself then $y$ is in $W$.* T

(viii) *Some orthogonal sets are not linearly independent.* F

**Problem 2.** *You are given the following vectors in* $\mathbb{R}^3$

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}.$$

(i) *(4 points) Argue that $\mathcal{B} = \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$!*

(ii) *(8 points )Determine the coordinate vector of $y$ relative to the basis $\mathcal{B}$. (in other words: find the vector $[y]_{\mathcal{B}}$).*

(i)
$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

3 pivots $\Rightarrow \{u_1, u_2, u_3\}$ is a base of $\mathbb{R}^3$

(ii) Need to solve the vector equation

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = y$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -1 & 1 & -1 & -4 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -1 & -5 \\ 0 & -1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -1 & -5 \\ 0 & -1 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & -1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{R_3 = 2R_3} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + \frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\rightarrow x_1 = 1; \; x_2 = -2; \; x_3 = 1 \Rightarrow \boxed{[y]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$ 2

**Problem 3.** *You are given the following matrix:*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

*(i) (8 points) Find a basis of RowA! What is* $\dim RowA$*?*

*(ii) (4 points) What is* $\dim ColA^T$*? How about* $RankA^T$*? How about* $\dim NulA^T$*? Argue your answer!*

(i) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_1]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - R_2 \\ R_4 = R_4 + R_2}]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[R_4 = \frac{1}{3} R_4]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A base of Row A is given by $\{[1; -1; 1]; [0; 1; 2]; [0; 0; 1]\}$

or

$\{[1; -1; 1]; [-1; 2; 1]; [0; -1; 1]\}$

dim Row A = 3, since there are 3 basis vectors

(ii) · Col $A^T$ is the same as Row A ; hence also $\dim \mathrm{Col}\, A^T = \dim \mathrm{Row} A = 3$

· By definition $\mathrm{Rank}\, A^T = \dim \mathrm{Col}\, A^T = 3$

· By the rank theorem $\mathrm{Rank}\, A^T + \dim \mathrm{Nul}\, A^T = 4$

number of columns of $A^T$

hence $\dim \mathrm{Nul}\, A^T = 1$.

**Problem 4.** *(12 points )Is the following matrix A diagonalizable? If yes, find the invertible matrix P and diagonal matrix D such that* $P^{-1}AP = D$.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

*Hint.* One of the roots of the characteristic equation should be very easy to guess. Using this information, a division of polynomials should help in finding all other roots. □

$1^{st}$ step find eigenvalues:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = (1-\lambda)(-5-\lambda)(1-\lambda) - 27 - 27 - 9(-5-\lambda) + 9(1-\lambda) + 9(1-\lambda) =$$

$$= (1-\lambda)(-5-\lambda)(1-\lambda) - 54 + 45 + 9\lambda + 18(1-\lambda)$$

$$= (1-\lambda)(-5-\lambda)(1-\lambda) + 9\lambda - 9 + 18(1-\lambda) =$$

[$(1-\lambda)$ is common factor]

$$= (1-\lambda)\left[(-5-\lambda)(1-\lambda) - 9 + 18\right] =$$

$$= (1-\lambda)(9 - 5 - \lambda + 5\lambda + \lambda^2)$$

$$= (1-\lambda)(4 + 4\lambda + \lambda^2) =$$

$$= (1-\lambda)(\lambda+2)^2 \rightarrow \lambda_1 = 1, \quad \lambda_2 = -2$$

$2^{nd}$ step: (eigenvectors)

$\boxed{\lambda_1 = 1}$ Find base of $\text{Nul}(A - I) = \text{Nul} \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \xrightarrow[R_2 = -R_2]{\text{divide all rows by } 3} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

solution is: $\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases}$ hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

eigenvector!

$\lambda_2 = -2$ Find base of $Nul(A+2I) = Nul \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[\text{rows by } 3]{\text{divide all}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[\substack{R_2 = R_2 + R_1 \\ R_3 = R_3 - R_1}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$\rightarrow$ solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigenvectors

We got 3 linearly independent eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}; \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

hence A is diagonalizable:

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P^{-1} A P = D$$

**Problem 5.** *You are given the following matrix:*

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

(i) *(8 points) Find the complex eigenvalues and basis of each eigenspace in $\mathbb{C}^2$ for the matrix $A$.*

(ii) *(4 points) If $A$ is diagonalizable over $\mathbb{C}$ then provide the complex valued invertible matrix $P$ and diagonal matrix $D$ such that $P^{-1}AP = D$.*

(i) eigenvalues: $0 = \det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1(-1) =$

$= (2-\lambda)^2 + 1$

$\Rightarrow (2-\lambda)^2 = -1 \Rightarrow 2-\lambda = \pm\sqrt{-1} = \pm i$

$\Rightarrow \boxed{\lambda_{1,2} = 2 \pm i}$

eigenvectors: $\boxed{\lambda_1 = 2+i}$ Find base of $\mathrm{Nul}(A-(2+i)I) = \mathrm{Nul}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow[R_1 = -R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & i & 0 \\ -i & 1 & 0 \end{array}\right] \xrightarrow[R_2 = R_2 + iR_1]{} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

basis eigenvector!

$\boxed{\lambda_2 = 2-i}$ As remarked in class we can find the corresponding basis eigenvector using conjugation: $\overline{\begin{bmatrix} -i \\ 1 \end{bmatrix}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

(ii) We found two linearly independent eigenvectors: $\begin{bmatrix} -i \\ 1 \end{bmatrix}; \begin{bmatrix} i \\ 1 \end{bmatrix}$, hence A is diagonalizable over $\mathbb{C}$: $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$

**Problem 6.** *You are given the following vectors in* $\mathbb{R}^4$*:*

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \; y = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 0 \end{bmatrix}.$$

*We introduce the subspace* $W = Span\{u_1, u_2, u_3\}$.

(i) *(4 points) Check that* $\mathcal{B} = \{u_1, u_2, u_3\}$ *is an orthogonal basis of* $W$*!*

(ii) *(8 points) Given that* $y$ *is in* $W$*, write* $y$ *as a linear combination of* $u_1, u_2, u_3$.

(i) $u_1 \cdot u_2 = 1 + 0 + -1 = 0$

$u_2 \cdot u_3 = 1 + 0 - 1 + 0 = 0$

$u_3 \cdot u_1 = 1 + 0 + 1 - 2 = 0$

$\rightarrow \{u_1; u_2; u_3\}$ is orthogonal. By a theorem we proved in class, orthogonal sets are linearly independent, hence $\{u_1; u_2; u_3\}$ is linearly independent. $\{u_1; u_2; u_3\}$ spans because by definition of $W$ we have

$$W = Span\{u_1; u_2; u_3\}$$

(ii) as $\{u_1; u_2; u_3\}$ is orthogonal, we may write:

$$y = \frac{u_1 \cdot y}{u_1 \cdot u_1} \cdot u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} \cdot u_2 + \frac{u_3 \cdot y}{u_3 \cdot u_3} \cdot u_3$$

$$= \frac{4 + 8 + 2}{1 + 4 + 1 + 1} \cdot u_1 + \frac{4 + 0 - 2 + 0}{1 + 1} \cdot u_2 + \frac{4 + 0 + 2 + 0}{1 + 1 + 4} \cdot u_3$$

$$\boxed{y = 2u_1 + u_2 + u_3}$$

**Problem 7.** *You are given the following vectors in* $\mathbb{R}^4$*:*

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix}.$$

*We introduce the subspace* $W = Span\{u_1, u_2, u_3\}$.

(i) (4 points) *Check that* $\mathcal{B} = \{u_1, u_2, u_3\}$ *is an orthogonal basis of* $W$*!*

(ii) (4 points) *Write* $y$ *as* $y = \hat{y} + z$*, where* $\hat{y}$ *is in* $W$ *and* $z$ *is in* $W^{\perp}$.

(iii) (4 points) *Find the distance of* $y$ *from the subspace* $W$*!*

(i) $u_1 \cdot u_2 = 1 + 0 + 0 - 1 = 0$

$v_1 \cdot u_3 = 1 - 2 + 0 + 1 = 0$

$u_2 \cdot u_3 = 1 + 0 + 0 - 1 = 0$

$\rightarrow \{v_1, u_2, u_3\}$ is orthogonal. By the same reasoning as in Problem 6(i) we have that $\{u_1, u_2, u_3\}$ is a basis of $W$.

(ii) We use the orthogonal decomposition theorem:

$$\hat{y} = \frac{u_1 \cdot y}{u_1 \cdot u_1} \cdot u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} \cdot u_2 + \frac{u_3 \cdot y}{u_3 \cdot u_3} \cdot u_3 =$$

$$= \frac{3-5-1}{1+1+1} \cdot u_1 + \frac{3+1}{2} u_2 + \frac{3+10-1}{1+4+1} u_3 =$$

$$= -1 \cdot u_1 + 2u_2 + 2u_3 = (-1)\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1+2+2 \\ 1 \quad +4 \\ 0 \\ -1-2+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ -1 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

(iii) $dist(y, W) = \|z\| = \sqrt{0^2 + 0^2 + 2^2 + 0^2} = 2$

**Problem 8.** *You are given the following vectors in* $\mathbb{R}^3$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(i) *(8 points) Using the Gram-Schmidt process turn the basis* $\{u_1, u_2, u_3\}$ *into an orthogonal basis* $\{v_1, v_2, v_3\}$.

(ii) *(4 points) Turn the orthogonal basis* $\{v_1, v_2, v_3\}$ *you just obtained into an ortho-normal basis* $\{w_1, w_2, w_3\}$.

(i) $$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

$$v_3 = u_3 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2 - \frac{1}{2}}{\frac{1}{4} + 4 + \frac{1}{4}} \cdot \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{3}{2}}{\frac{9}{2}} \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

(ii) $$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\frac{1}{4} + 4 + \frac{1}{4}}} \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{\frac{18}{4}}} \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} = \frac{2}{3\sqrt{2}} \begin{bmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}}} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$