Endoscopic Transfer of the Bernstein Center

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Outline

Overview

- The “fundamental lemma” (FL) is now a theorem: Ngô, Waldspurger, Hales,...
- along with its twisted, weighted, and twisted-weighted versions (Ngô, Waldspurger, Laumon-Chaudouard...).
- These play a crucial role in automorphic forms (Langlands functoriality via comparison of trace formulas)
- and in arithmetic via cohomology of Shimura varieties, and their Hasse-Weil zeta functions.

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There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.

It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.

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Groups

G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

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Hierarchy: *G split $\subset$ unramified $\subset$ quasi-split $\subset$ ....*

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

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Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
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Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

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Endoscopic groups: stabilization of the trace formula (**bold** = global).

- Trace formula. Let $\mathbf{f} = \otimes_v f_v \in C_c^\infty(\mathbf{G}(\mathbb{A}))$. Roughly, the Trace Formula is an equality of the form $T_{\rm geom}(\mathbf{f}) = T_{\rm spec}(\mathbf{f})$

$$\sum_{\gamma_0 \in \mathbf{G}(\mathbb{Q})/\sim} c_{\gamma_0}\, \mathrm{O}_{\gamma_0}^{\mathbf{G}(\mathbb{A})}(\mathbf{f}) + \cdots = \sum_{\pi = \otimes'_v \pi_v} m(\pi)\, \mathrm{tr}\, \pi(\mathbf{f}) + \cdots .$$

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$$T_*^{\mathbf{G}}(\mathbf{f}) = \sum_{\mathbf{H}/\sim} \iota(\mathbf{G}, \mathbf{H})\, ST_*^{\mathbf{H}}(\mathbf{f}^{\mathbf{H}}),$$

where $* \in \{\text{“geom”}, \text{“spec, disc”}\}$.

- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^pK_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\, H_c^\bullet(Sh)) = \sum_{\gamma_0; \gamma, \delta} c(\gamma_0; \gamma, \delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

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- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^p K_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\ H_c^\bullet(Sh)) = \sum_{\gamma_0;\gamma,\delta} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

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Can be local or global. Consider local case.

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- H connected reductive, quasisplit over F
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Examples

$G = \mathrm{GL}_n$: $H =$ Levi subgroup of GL_n

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$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

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$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

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$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  The spherical base change homomorphism $b_r : \mathcal{H}(G_r, K_r) \rightarrow \mathcal{H}(G, K)$ produces matching pairs: for all f_r, we have $f_r \leftrightarrow b_r(f_r)$.
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Remark: Formally B is a special case of C, but B is used in proof of C.

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  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

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3 examples (spherical)

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The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in R(\mathrm{GL}_n(F)) \mapsto \Pi \in R(\mathrm{GL}_n(F_r))$.
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B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

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Waldspurger's (twisted) transfer theorem

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Let H be any Frobenius-twisted endoscopic group for G_r, θ. Given $f_r \in \mathcal{H}(G_r)$, there exists at least one $f^H \in \mathcal{H}(H)$ with

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- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T,G) \cong S_2$.

Set $\mathcal{Z}(G,I) = Z(\mathcal{H}(G,I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G,I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

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The Cartan decomposition gives a basis of spherical functions $f_\mu = 1_{K\mu(\varpi)K}$, (one for each dominant cocharacter $\mu \in X_*(T)$).

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Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}^{G}_{N_r\delta}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

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Goal

Generalize the transfer homomorphism $b_r : \mathcal{H}(G_r, K_r) \rightarrow \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us (conjecturally) to construct many matching pairs $f_r \leftrightarrow f^H$.

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4 views of the Bernstein center

First step: construct many elements of the Bernstein center.

The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ – regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting σ. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$.
Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

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- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

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$$\mathrm{Rep}(\widehat{G}) \to \mathcal{Z}(G).$$

Two reasons why:

1) The test functions in $\mathcal{Z}(G)$ needed for Shimura varieties all come from $\mathrm{Rep}(\widehat{G})$.

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That is, is there a canonical map, making the diagram commute?

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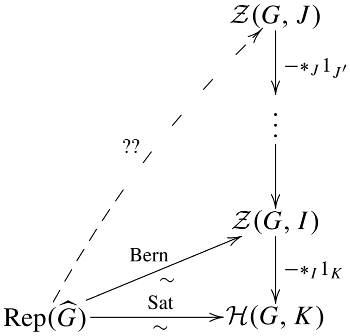

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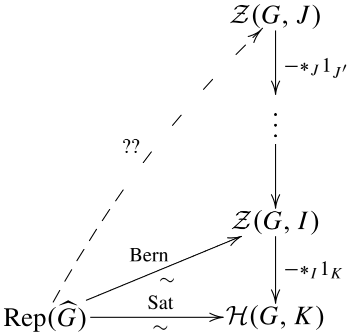

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Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^LG = \widehat{G} \rtimes W_F$ with usual properties, and

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Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^LG)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \ \mapsto \ Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

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Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

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Grothendieck quasi-unipotence: $\exists$ finite-index subgroup of I_F acting purely unipotently on V.

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$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I)$.

Aside: semisimple L-functions

To

$$W_{\mathbb{Q}_p} \xrightarrow{\varphi_\pi} {}^L G \xrightarrow{r_V} \mathrm{Aut}(V)$$

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$$\log(L^{ss}(s, \pi, r_V)) = \sum_{r \geq 1} \mathrm{tr}^{ss}(\varphi_\pi(\Phi^r), V) \frac{p^{-rs}}{r}.$$

Importance: conjecturally we can express $Z_{\mathfrak{p}}^{ss}(s, Sh)$ in terms of several $L^{ss}(s-?, \pi_p, r)$, where r is determined by Sh.

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Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \to {}^LG$ and $\varphi_\sigma : W_F \to {}^LM \hookrightarrow {}^LG$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

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The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.
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Aside: explicit matching pairs

With the FL and Waldspurger's transfer now available, it makes sense to try to find explicit matching pairs $f_r \leftrightarrow f^H$ where f belongs to a prescribed family.

Example: Kazhdan-Varshavsky: f and f^H are Deligne-Lusztig functions coming from the representation theory of finite groups of Lie type.

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Results: Base change fundamental lemmas – essentially Iwahori level

Consider $H = G$, i.e., $f_r \leftrightarrow f$ has the sense of stable base change.

Theorem (H. 2009, 2010 – Predecessors of Ztransfer conjecture)

For G unramified over F, and $J = I, I^+$, or parahoric, there exists a base-change homomorphism

$$b_r : \mathcal{Z}(G_r, J_r) \to \mathcal{Z}(G, J)$$

defined explicitly in terms of the Bernstein isomorphism (on the dual side) with the property

$$f_r \leftrightarrow b_r(f_r).$$

This was used to study certain Shimura varieties with parahoric or $\Gamma_1(p)$-level structure at p (see below).

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Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow C \cdot 1_{J_H}$ (C an explicit constant).

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

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Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

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Conjecture (H.-Kottwitz – "clean form")

In the situation above, for every $r \geq 1$, the test function $f_{r,1}$ above satisfies: the alternating sum of the semi-simple traces

$$\sum_{i=0}^{2\dim(Sh)} (-1)^i \mathrm{tr}^{ss}(\Phi_{\mathfrak{p}}^r, \mathrm{H}^i(Sh \times_E \bar{E}_{\mathfrak{p}}, \bar{\mathbb{Q}}_\ell))$$

equals the trace

$$\mathrm{tr}(1_{K^p} \otimes f_{r,1} \otimes f_\infty, \mathrm{L}^2(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^\circ \backslash \mathbf{G}(\mathbb{A}))).$$

In fact, this is really a consequence of the "real" Test Function Conjecture together with the Ztransfer conjecture.

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Conjecture (H.-Kottwitz – “clean form”)

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$$\sum_{i=0}^{2\dim(Sh)} (-1)^i \operatorname{tr}^{ss}(\Phi_{\mathfrak{p}}^r, \mathrm{H}^i(Sh \times_E \bar{E}_{\mathfrak{p}}, \bar{\mathbb{Q}}_\ell))$$

equals the trace

$$\operatorname{tr}(1_{K^p} \otimes f_{r,1} \otimes f_\infty, \mathrm{L}^2(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^\circ \backslash \mathbf{G}(\mathbb{A}))).$$

In fact, this is really a consequence of the “real” Test Function Conjecture together with the Ztransfer conjecture.

Consequence: Automorphy of local factors of Hasse-Weil Zeta functions

Remark

When the above equation holds (e.g. in certain "nice" cases of "unitary" Shimura varieties as above when both Ztransfer conjecture and "real" test function conjecture are known), we have

$$Z^{ss}_{\mathfrak{p}}(s, Sh) = \prod_{\pi_f} L^{ss}(s - \frac{\dim Sh}{2}, \pi_p, r_{\mu^*})^{n(\pi_f)} \quad (1)$$

where $\pi_f = \pi^{p,\infty} \otimes \pi_p$ *ranges over certain representations of* $\mathbf{G}(\mathbb{A}_f)$ *and* $n(\pi_f) \in \mathbb{Z}$.

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ is split of type A or C and K_p is parahoric.

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A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$), if $K_p = I^+$.

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Summary

The usual statements of the FL and transfer homomorphisms fit into a larger framework, defined on part of the Bernstein center in an explicit way.

In certain cases, the construction is unconditional, and sometimes can be proved.

For GSp_{2n}, there is hope it can be established and used to study the associated non-compact Shimura varieties with bad reduction.

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