Endoscopic Transfer of the Bernstein Center

Thomas J. Haines

Mathematics Department
University of Maryland

April 14, 2011

Outline

Overview

- The "fundamental lemma" (FL) is now a theorem: Ngô, Waldspurger, Hales,...
- along with its twisted, weighted, and twisted-weighted versions (Ngô, Waldspurger, Laumon-Chaudouard...).
- These play a crucial role in automorphic forms (Langlands functoriality via comparison of trace formulas)
- and in arithmetic via cohomology of Shimura varieties, and their Hasse-Weil zeta functions.

Overview

- The "fundamental lemma" (FL) is now a theorem: Ngô, Waldspurger, Hales,...
- along with its twisted, weighted, and twisted-weighted versions (Ngô, Waldspurger, Laumon-Chaudouard...).
- These play a crucial role in automorphic forms (Langlands functoriality via comparison of trace formulas)
- and in arithmetic via cohomology of Shimura varieties, and their Hasse-Weil zeta functions.

Overview

- The "fundamental lemma" (FL) is now a theorem: Ngô, Waldspurger, Hales,...
- along with its twisted, weighted, and twisted-weighted versions (Ngô, Waldspurger, Laumon-Chaudouard...).
- These play a crucial role in automorphic forms (Langlands functoriality via comparison of trace formulas)
- and in arithmetic via cohomology of Shimura varieties, and their Hasse-Weil zeta functions.

Overview

- The “fundamental lemma” (FL) is now a theorem: Ngô, Waldspurger, Hales,...
- along with its twisted, weighted, and twisted-weighted versions (Ngô, Waldspurger, Laumon-Chaudouard...).
- These play a crucial role in automorphic forms (Langlands functoriality via comparison of trace formulas)
- and in arithmetic via cohomology of Shimura varieties, and their Hasse-Weil zeta functions.

There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.

It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.

(Another approach to bad reduction is that of Harris-Taylor, continued by Fargues, Mantovan, Shin,...)

There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.

It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.

(Another approach to bad reduction is that of Harris-Taylor, continued by Fargues, Mantovan, Shin,...)

There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.

It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.

(Another approach to bad reduction is that of Harris-Taylor, continued by Fargues, Mantovan, Shin,...)

There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.

It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.

(Another approach to bad reduction is that of Harris-Taylor, continued by Fargues, Mantovan, Shin,...)

There exist variants, more recently proved:

- FL of Jacquet-Ye (Ngô, Jacquet)
- FL conjectured by Jacquet-Rallis (Z. Yun, J. Gordon)

Purpose of talk: discuss a new conjectural variant related to the Bernstein center.
It has applications to Shimura varieties with **bad reduction** (and arose there), extending to bad reduction cases the Langlands-Kottwitz approach to Shimura varieties with good reduction.
(Another approach to bad reduction is that of Harris-Taylor, continued by Fargues, Mantovan, Shin,...)

Groups

G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: *G split $\subset$ unramified $\subset$ quasi-split $\subset$ ....*

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Groups

G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: *G split $\subset$ unramified $\subset$ quasi-split $\subset$ ....*

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Groups

G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: G *split* $\subset$ *unramified* $\subset$ *quasi-split* $\subset \dots$.

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Groups
G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: *G split $\subset$ unramified $\subset$ quasi-split $\subset \ldots$.*

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Groups
G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: G *split* $\subset$ *unramified* $\subset$ *quasi-split* $\subset \dots$.

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Groups

G will be a **connected reductive group** over some field.

Examples: GL_n, SL_n, SO_{2n+1}, Sp_{2n}, G_2, E_8.

We need the **Langlands dual group** $\widehat{G} = \widehat{G}(\mathbb{C})$, defined to have dual root data.

Examples: GL_n, PGL_n, Sp_{2n}, SO_{2n+1}, G_2, E_8.

Hierarchy: G *split* $\subset$ *unramified* $\subset$ *quasi-split* $\subset \ldots$.

In this talk we often assume: G split over F, and $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

"Frobenius Twisted versions": F_r/F unramified, $\langle \theta \rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$O_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma \backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(x^{-1}\delta\theta(x))\,dx$.
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- Spectral analogues: $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi(\theta)(f_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K\backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma\backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\,dg$.
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$
- *Spectral analogues:* $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi(f_r)$

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^{\infty}(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma \backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\, d\bar{g}$.
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- *Spectral analogues:* $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi(f_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K \backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma\backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\, d\bar{g}$,
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- *Spectral analogues*: $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi\theta(f_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K\backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma\backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\, d\bar{g}$,
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- *Spectral analogues*: $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi\theta(f_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K\backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma\backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\, d\bar{g}$,
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- *Spectral analogues*: $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi\theta(f_r)$.

Some distributions on Hecke algebras

- p-adic field $F \supset \mathcal{O} \ni \varpi$. $\Gamma = \mathrm{Gal}(\bar{F}/F)$.
- If G defined over F, often write $G = G(F)$. Set $K := G(\mathcal{O})$, when $G/\mathcal{O}$.
- **Hecke algebra** $\mathcal{H}(G) = C_c^\infty(G(F), \mathbb{C})$.
- **spherical Hecke algebra** $\mathcal{H}(G, K) = C_c(K\backslash G(F)/K)$.

Let $\gamma \in G(F)$ be regular semisimple (i.e. G_γ a maximal torus), $f \in \mathcal{H}(G)$.

- $$\mathrm{O}_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f.$$
- $$\mathrm{SO}_\gamma(f) := \int_{(G_\gamma\backslash G)(F)} f = \sum_{\gamma' \overset{st}{\sim} \gamma} \mathrm{O}_{\gamma'}(f).$$

"Frobenius Twisted versions": F_r/F unramified, $\langle\theta\rangle = \mathrm{Gal}(F_r/F)$, $\delta \in G_r := G(F_r)$ s.t. $N_r\delta := \delta\theta(\delta)\cdots\theta^{r-1}(\delta)$ regular semisimple. Let $f_r \in \mathcal{H}(G_r)$.

- $\mathrm{TO}_{\delta\theta}(f_r) = \int_{G_{\delta\theta}\backslash G_r} f_r(g^{-1}\delta\theta(g))\, d\bar{g}$,
- $\mathrm{SO}_{\delta\theta}(f_r) = \cdots$.
- *Spectral analogues*: $f \mapsto \mathrm{tr}\,\pi(f)$, and $f_r \mapsto \mathrm{tr}\,\Pi\theta(f_r)$.

Endoscopic groups: stabilization of the trace formula (**bold** = global).

- Trace formula. Let $\mathbf{f} = \otimes_v f_v \in C_c^\infty(\mathbf{G}(\mathbb{A}))$. Roughly, the Trace Formula is an equality of the form $T_{\rm geom}(\mathbf{f}) = T_{\rm spec}(\mathbf{f})$

$$\sum_{\gamma_0 \in \mathbf{G}(\mathbb{Q})/\sim} c_{\gamma_0}\, \mathrm{O}_{\gamma_0}^{\mathbf{G}(\mathbb{A})}(\mathbf{f}) + \cdots = \sum_{\pi = \otimes'_v \pi_v} m(\pi)\, \mathrm{tr}\, \pi(\mathbf{f}) + \cdots .$$

- Roughly, the stabilization of the trace formula is an expression

$$T_*^{\mathbf{G}}(\mathbf{f}) = \sum_{\mathbf{H}/\sim} \iota(\mathbf{G}, \mathbf{H})\, ST_*^{\mathbf{H}}(\mathbf{f}^{\mathbf{H}}),$$

where $* \in \{\text{“geom”}, \text{“spec, disc”}\}$.

- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^pK_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\, H_c^\bullet(Sh)) = \sum_{\gamma_0;\gamma,\delta} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

Endoscopic groups: stabilization of the trace formula (**bold** = global).

- Trace formula. Let $\mathbf{f} = \otimes_v f_v \in C_c^\infty(\mathbf{G}(\mathbb{A}))$. Roughly, the Trace Formula is an equality of the form $T_{\text{geom}}(\mathbf{f}) = T_{\text{spec}}(\mathbf{f})$

$$\sum_{\gamma_0 \in \mathbf{G}(\mathbb{Q})/\sim} c_{\gamma_0}\, \mathrm{O}_{\gamma_0}^{\mathbf{G}(\mathbb{A})}(\mathbf{f}) + \cdots = \sum_{\pi = \otimes'_v \pi_v} m(\pi)\, \mathrm{tr}\, \pi(\mathbf{f}) + \cdots .$$

- Roughly, the stabilization of the trace formula is an expression

$$T_*^{\mathbf{G}}(\mathbf{f}) = \sum_{\mathbf{H}/\sim} \iota(\mathbf{G}, \mathbf{H})\, ST_*^{\mathbf{H}}(\mathbf{f}^{\mathbf{H}}),$$

where $* \in \{\text{“geom”}, \text{“spec, disc”}\}$.

- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^p K_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\, H_c^\bullet(Sh)) = \sum_{\gamma_0;\gamma,\delta} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

Endoscopic groups: stabilization of the trace formula (**bold** = global).

- Trace formula. Let $\mathbf{f} = \otimes_v f_v \in C_c^\infty(\mathbf{G}(\mathbb{A}))$. Roughly, the Trace Formula is an equality of the form $T_{\mathrm{geom}}(\mathbf{f}) = T_{\mathrm{spec}}(\mathbf{f})$

$$\sum_{\gamma_0 \in \mathbf{G}(\mathbb{Q})/\sim} c_{\gamma_0}\, \mathrm{O}_{\gamma_0}^{\mathbf{G}(\mathbb{A})}(\mathbf{f}) + \cdots = \sum_{\pi = \otimes'_v \pi_v} m(\pi)\, \mathrm{tr}\, \pi(\mathbf{f}) + \cdots .$$

- Roughly, the stabilization of the trace formula is an expression

$$T_*^{\mathbf{G}}(\mathbf{f}) = \sum_{\mathbf{H}/\sim} \iota(\mathbf{G}, \mathbf{H})\, ST_*^{\mathbf{H}}(\mathbf{f}^{\mathbf{H}}),$$

where $* \in \{\text{“geom”}, \text{“spec, disc”}\}$.

- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^p K_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\ H_c^\bullet(Sh)) = \sum_{\gamma_0;\gamma,\delta} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

Endoscopic groups: stabilization of the trace formula (**bold** = global).

- Trace formula. Let $\mathbf{f} = \otimes_v f_v \in C_c^\infty(\mathbf{G}(\mathbb{A}))$. Roughly, the Trace Formula is an equality of the form $T_{\rm geom}(\mathbf{f}) = T_{\rm spec}(\mathbf{f})$

$$\sum_{\gamma_0 \in \mathbf{G}(\mathbb{Q})/\sim} c_{\gamma_0}\, \mathrm{O}_{\gamma_0}^{\mathbf{G}(\mathbb{A})}(\mathbf{f}) + \cdots = \sum_{\pi = \otimes'_v \pi_v} m(\pi)\, \mathrm{tr}\, \pi(\mathbf{f}) + \cdots .$$

- Roughly, the stabilization of the trace formula is an expression

$$T_*^{\mathbf{G}}(\mathbf{f}) = \sum_{\mathbf{H}/\sim} \iota(\mathbf{G}, \mathbf{H})\, ST_*^{\mathbf{H}}(\mathbf{f}^{\mathbf{H}}),$$

  where $* \in \{\text{“geom''}, \text{“spec, disc''}\}$.
- Endoscopic groups enter into the pseudostabilization of the Lefschetz formula for Shimura varieties $Sh = Sh(\mathbf{G}, X, K^pK_p)$

$$\mathrm{Lef}(\Phi_{\mathfrak{p}}^r;\ H_c^\bullet(Sh)) = \sum_{\gamma_0;\gamma,\delta} c(\gamma_0;\gamma,\delta)\, \mathrm{O}_\gamma^{\mathbf{G}(\mathbb{A}^{p,\infty})}(f^p)\, \mathrm{TO}_{\delta\theta}^{G(\mathbb{Q}_{p^r})}(\phi_r).$$

Endoscopic groups: What are they?

Can be local or global. Consider local case.

An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\, \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$
- $W_H \subset W_G$, so $\widehat{T}_H / W_H \to \widehat{T}_G / W_G$

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.

I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?
Can be local or global. Consider local case.
An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} \mathcal{H}(G,K) & & \overset{b}{\dashrightarrow} & & \mathcal{H}(H,K_H) & & \\ \downarrow \wr\ \mathrm{Sat} & & & & \downarrow \wr\ \mathrm{Sat} & & \\ \mathrm{Rep}(\widehat{G}) = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.
I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?

Can be local or global. Consider local case.

An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^{\circ}$, some $s \in Z(\widehat{H})^{\Gamma}$.
- $((H, s, \eta)$ taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H / W_H \to \widehat{T}_G / W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} \mathcal{H}(G,K) & & \overset{b}{\dashrightarrow} & & \mathcal{H}(H,K_H) & & \\ \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \\ \mathrm{Rep}(\widehat{G}) = \mathbb{C}[\widehat{T}_G/W_G] & & \longrightarrow & & \mathbb{C}[\widehat{T}_H/W_H] = \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.

I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?
Can be local or global. Consider local case.
An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \widetilde{\to} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} \mathcal{H}(G,K) & & \overset{b}{\dashrightarrow} & & \mathcal{H}(H,K_H) & & \\ \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \\ \mathrm{Rep}(\widehat{G}) = \mathbb{C}[\widehat{T}_G/W_G] & & \longrightarrow & & \mathbb{C}[\widehat{T}_H/W_H] = \mathrm{Rep}(\widehat{H}). & & \end{array}$$

We will generalize this picture.
I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?
Can be local or global. Consider local case.
An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \downarrow{\scriptstyle \mathrm{Sat}\,\wr} & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.
I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?

Can be local or global. Consider local case.

An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- $((H, s, \eta)$ taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H / W_H \rightarrow \widehat{T}_G / W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \rightarrow \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & \downarrow \mathrm{Sat}\ \wr & & \downarrow \mathrm{Sat}\ \wr & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.

Endoscopic groups: What are they?
Can be local or global. Consider local case.
An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \overset{\sim}{\to} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & \downarrow{\scriptstyle \mathrm{Sat}\ \wr} & & \downarrow{\scriptstyle \mathrm{Sat}\ \wr} & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.
I will not define (Frobenius) twisted endoscopic groups

Endoscopic groups: What are they?

Can be local or global. Consider local case.

An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & \mathrm{Sat} \downarrow \wr & & \mathrm{Sat} \downarrow \wr & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.

I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?

Can be local or global. Consider local case.

An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^{\circ}$, some $s \in Z(\widehat{H})^{\Gamma}$.
- ((H, s, η) taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H / W_H \rightarrow \widehat{T}_G / W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \rightarrow \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc}
 & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\
 & & \mathrm{Sat}\downarrow\wr & & \mathrm{Sat}\downarrow\wr & & \\
\mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}).
\end{array}$$

We will generalize this picture.

I will not define (Frobenius) twisted endoscopic groups.

Endoscopic groups: What are they?
Can be local or global. Consider local case.
An (standard) endoscopic group H for G satisfies

- H connected reductive, quasisplit over F
- $\exists\ \eta : \widehat{H} \hookrightarrow \widehat{G}$ such that $\eta : \widehat{H} \cong C_{\widehat{G}}(\eta(s))^\circ$, some $s \in Z(\widehat{H})^\Gamma$.
- $((H, s, \eta)$ taken up to an equivalence relation.)
- H, G share a Cartan over F: there exists $T_H \xrightarrow{\sim} T_G \subset G$.
- $W_H \subset W_G$, so $\widehat{T}_H/W_H \to \widehat{T}_G/W_G$.

There is a transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$. If G and H split:

$$\begin{array}{ccccccc} & & \mathcal{H}(G,K) & \overset{b}{\dashrightarrow} & \mathcal{H}(H,K_H) & & \\ & & {\scriptstyle \mathrm{Sat}}\downarrow\wr & & {\scriptstyle \mathrm{Sat}}\downarrow\wr & & \\ \mathrm{Rep}(\widehat{G}) & = & \mathbb{C}[\widehat{T}_G/W_G] & \longrightarrow & \mathbb{C}[\widehat{T}_H/W_H] & = & \mathrm{Rep}(\widehat{H}). \end{array}$$

We will generalize this picture.
I will not define (Frobenius) twisted endoscopic groups.

Examples

$G = \mathrm{GL}_n$: $H =$ Levi subgroup of GL_n

$G = \mathrm{PGL}_2$: $H = \mathrm{PGL}_2$ or GL_1.

$G = \mathrm{SL}_2$: $H = \mathrm{SL}_2, \mathrm{GL}_1$, or $U^1_{E/F}$, E/F quadratic.

$G = G_2$: $H = G_2, \mathrm{SO}_4, \mathrm{PGL}_3$, or $\mathrm{GL}_1 \times \mathrm{GL}_1$.

Examples

$G = \mathrm{GL}_n$: $H =$ Levi subgroup of GL_n

$G = \mathrm{PGL}_2$: $H = \mathrm{PGL}_2$ or GL_1.

$G = \mathrm{SL}_2$: $H = \mathrm{SL}_2, \mathrm{GL}_1$, or $U^1_{E/F}$, E/F quadratic.

$G = G_2$: $H = G_2, \mathrm{SO}_4, \mathrm{PGL}_3$, or $\mathrm{GL}_1 \times \mathrm{GL}_1$.

Examples

$G = \mathrm{GL}_n$: $H =$ Levi subgroup of GL_n

$G = \mathrm{PGL}_2$: $H = \mathrm{PGL}_2$ or GL_1.

$G = \mathrm{SL}_2$: $H = \mathrm{SL}_2, \mathrm{GL}_1$, or $U^1_{E/F}$, E/F quadratic.

$G = G_2$: $H = G_2, \mathrm{SO}_4, \mathrm{PGL}_3$, or $\mathrm{GL}_1 \times \mathrm{GL}_1$.

Examples

$G = \mathrm{GL}_n$: $H =$ Levi subgroup of GL_n

$G = \mathrm{PGL}_2$: $H = \mathrm{PGL}_2$ or GL_1.

$G = \mathrm{SL}_2$: $H = \mathrm{SL}_2, \mathrm{GL}_1$, or $U^1_{E/F}$, E/F quadratic.

$G = G_2$: $H = G_2, \mathrm{SO}_4, \mathrm{PGL}_3$, or $\mathrm{GL}_1 \times \mathrm{GL}_1$.

Matching functions H endoscopic group for G.
$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

Definition

$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\delta \in G(F_r)/\theta-conj} \Delta(\gamma_H, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r).$$

Here $\Delta(\gamma_H, \delta) \in \mathbb{C}^\times$ are the transfer factors associated to H and G_r.

The “Frobenius twist” is built into them.

When $r = 1$ (no Frobenius twist), we get the standard transfer factors.

Matching functions H endoscopic group for G.

$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

Definition

$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\delta \in G(F_r)/\theta-\text{conj}} \Delta(\gamma_H, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r).$$

Here $\Delta(\gamma_H, \delta) \in \mathbb{C}^\times$ are the transfer factors associated to H and G_r.

The "Frobenius twist" is built into them.

When $r = 1$ (no Frobenius twist), we get the standard transfer factors.

Matching functions H endoscopic group for G.

$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

Definition

$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\delta \in G(F_r)/\theta-\mathit{conj}} \Delta(\gamma_H, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r).$$

Here $\Delta(\gamma_H, \delta) \in \mathbb{C}^\times$ are the transfer factors associated to H and G_r.

The “Frobenius twist” is built into them.

When $r = 1$ (no Frobenius twist), we get the standard transfer factors.

Matching functions H endoscopic group for G.

$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

Definition

$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\delta \in G(F_r)/\theta-\mathit{conj}} \Delta(\gamma_H, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r).$$

Here $\Delta(\gamma_H, \delta) \in \mathbb{C}^\times$ are the transfer factors associated to H and G_r.

The “Frobenius twist” is built into them.

When $r = 1$ (no Frobenius twist), we get the standard transfer factors.

Matching functions H endoscopic group for G.

$f_r \in \mathcal{H}(G_r)$, $f^H \in \mathcal{H}(H)$.

Definition

$f_r \leftrightarrow f^H$ *if, for all* $\gamma_H \in H^{G-\mathrm{sr}}(F)$,

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\delta \in G(F_r)/\theta-\textit{conj}} \Delta(\gamma_H, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r).$$

Here $\Delta(\gamma_H, \delta) \in \mathbb{C}^\times$ are the transfer factors associated to H and G_r.

The "Frobenius twist" is built into them.

When $r = 1$ (no Frobenius twist), we get the standard transfer factors.

Special cases $f_r \leftrightarrow f^H$:

- Stable Base Change: $H = G$. Set $f := f^H$ and $\gamma := \gamma_H$.

$$\mathrm{SO}^G_\gamma(f) = \sum_{\delta \in G_r/\theta-\text{conj}} \Delta(\gamma, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r),$$

where

$$\Delta(\gamma, \delta) = \begin{cases} 1, & N_r(\delta) \overset{st}{\sim} \gamma \\ 0, & \text{otherwise.} \end{cases}.$$

- Standard endoscopy: $r = 1$, i.e. $G_r = G$. Set $f := f_r$ and $\gamma := \delta$.

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\gamma \in G(F)/\text{conj}} \Delta(\gamma_H, \gamma)\, \mathrm{O}^G_\gamma(f).$$

Langlands-Shelstad transfer factors $\Delta(\gamma_H, \gamma)$ much harder to define.

Special cases $f_r \leftrightarrow f^H$:

- Stable Base Change: $H = G$. Set $f := f^H$ and $\gamma := \gamma_H$.

$$\mathrm{SO}^G_\gamma(f) = \sum_{\delta \in G_r/\theta-\text{conj}} \Delta(\gamma, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r),$$

where

$$\Delta(\gamma, \delta) = \begin{cases} 1, & N_r(\delta) \overset{st}{\sim} \gamma \\ 0, & \text{otherwise.} \end{cases}$$

- Standard endoscopy: $r = 1$,i.e. $G_r = G$. Set $f := f_r$, and $\gamma := \delta$.

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\gamma \in G(F)/\mathrm{conj}} \Delta(\gamma_H, \gamma)\, \mathrm{O}^G_\gamma(f).$$

Langlands-Shelstad transfer factors $\Delta(\gamma_H, \gamma)$ much harder to define.

Special cases $f_r \leftrightarrow f^H$:

- Stable Base Change: $H = G$. Set $f := f^H$ and $\gamma := \gamma_H$.

$$\mathrm{SO}^G_\gamma(f) = \sum_{\delta \in G_r/\theta-\text{conj}} \Delta(\gamma, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r),$$

where

$$\Delta(\gamma, \delta) = \begin{cases} 1, & N_r(\delta) \overset{st}{\sim} \gamma \\ 0, & \text{otherwise.} \end{cases}$$

- Standard endoscopy: $r = 1$,i.e. $G_r = G$. Set $f := f_r$, and $\gamma := \delta$.

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\gamma \in G(F)/\text{conj}} \Delta(\gamma_H, \gamma)\, \mathrm{O}^G_\gamma(f).$$

Langlands-Shelstad transfer factors $\Delta(\gamma_H, \gamma)$ much harder to define.

Special cases $f_r \leftrightarrow f^H$:

- Stable Base Change: $H = G$. Set $f := f^H$ and $\gamma := \gamma_H$.

$$\mathrm{SO}^G_\gamma(f) = \sum_{\delta \in G_r/\theta-\text{conj}} \Delta(\gamma, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r),$$

where

$$\Delta(\gamma, \delta) = \begin{cases} 1, & N_r(\delta) \overset{st}{\sim} \gamma \\ 0, & \text{otherwise.} \end{cases}$$

- Standard endoscopy: $r = 1$,i.e. $G_r = G$. Set $f := f_r$, and $\gamma := \delta$.

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\gamma \in G(F)/\text{conj}} \Delta(\gamma_H, \gamma)\, \mathrm{O}^G_\gamma(f).$$

Langlands-Shelstad transfer factors $\Delta(\gamma_H, \gamma)$ much harder to define.

Special cases $f_r \leftrightarrow f^H$:

- Stable Base Change: $H = G$. Set $f := f^H$ and $\gamma := \gamma_H$.

$$\mathrm{SO}^G_\gamma(f) = \sum_{\delta \in G_r/\theta-\text{conj}} \Delta(\gamma, \delta)\, \mathrm{TO}^{G_r}_{\delta\theta}(f_r),$$

  where

$$\Delta(\gamma, \delta) = \begin{cases} 1, & N_r(\delta) \overset{st}{\sim} \gamma \\ 0, & \text{otherwise.} \end{cases}$$

- Standard endoscopy: $r = 1$,i.e. $G_r = G$. Set $f := f_r$, and $\gamma := \delta$.

$$\mathrm{SO}^H_{\gamma_H}(f^H) = \sum_{\gamma \in G(F)/\text{conj}} \Delta(\gamma_H, \gamma)\, \mathrm{O}^G_\gamma(f).$$

  Langlands-Shelstad transfer factors $\Delta(\gamma_H, \gamma)$ much harder to define.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists b_r : \mathcal{H}(G_r, K_r) \to \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.

- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.

- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists b_r : \mathcal{H}(G_r, K_r) \rightarrow \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow h(f)$, where h is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists\, b_r : \mathcal{H}(G_r, K_r) \to \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists\, b_r : \mathcal{H}(G_r, K_r) \rightarrow \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists b_r : \mathcal{H}(G_r, K_r) \to \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists\, b_r : \mathcal{H}(G_r, K_r) \rightarrow \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

3 examples (spherical)

$K = G(\mathcal{O})$ (hyperspecial), with analogue $K_r \subset G_r$.

- Example A: BCFL.
  Via Satake, $\exists b_r : \mathcal{H}(G_r, K_r) \to \mathcal{H}(G, K)$ s.t.
  $f_r \leftrightarrow b_r(f_r)$ in sense of stable base change.
- Example B: FL: $K = G(\mathcal{O})$, $K_H = H(\mathcal{O})$. Then $1_K \leftrightarrow 1_{K_H}$ in the sense of standard endoscopy.
- Example C: Hales' spherical transfer.
  $f \leftrightarrow b(f)$, where b is the (spherical) transfer homomorphism above.

Remark: Formally B is a special case of C, but B is used in proof of C.

Here there is a natural map of Hecke algebras producing matching pairs.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse; used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction.

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

The roles they played

A. came historically first.

- GL_2 Langlands, GL_3 Kottwitz
- GL_n Arthur-Clozel: $f_r \leftrightarrow b_r(f_r)$ dual to and used in local lifting $\pi \in \mathcal{R}(\mathrm{GL}_n(F)) \mapsto \Pi \in \mathcal{R}(\mathrm{GL}_n(F_r))$.
- G *unramified*: unit elements: Kottwitz
- general elements: Clozel, Labesse: used by Kottwitz for Shimura varieties with good reduction...

B. is essential for everything: Stabilization of trace formula, Shimura varieties, and Waldspurger's transfer theorem (more later).

C: provided a step in FL: Global-local argument used by Hales to show FL for a.e. p implies FL for all p.

Waldspurger's (twisted) transfer theorem

Theorem

Given $f_r \in \mathcal{H}(G_r)$, there exists at least one $f^H \in \mathcal{H}(H)$ with

$$f_r \leftrightarrow f^H.$$

However, the correspondence $f_r \mapsto f^H$ is not given by a natural geometric rule on the dual side, i.e. it is not (a priori) *spectrally explicit*.

Waldspurger's (twisted) transfer theorem

Theorem

Given $f_r \in \mathcal{H}(G_r)$, there exists at least one $f^H \in \mathcal{H}(H)$ with

$$f_r \leftrightarrow f^H.$$

However, the correspondence $f_r \mapsto f^H$ is not given by a natural geometric rule on the dual side, i.e. it is not (a priori) *spectrally explicit*.

Waldspurger's (twisted) transfer theorem

Theorem

Given $f_r \in \mathcal{H}(G_r)$, there exists at least one $f^H \in \mathcal{H}(H)$ with

$$f_r \leftrightarrow f^H.$$

However, the correspondence $f_r \mapsto f^H$ is not given by a natural geometric rule on the dual side, i.e. it is not (a priori) *spectrally explicit*.

Transition to Bernstein center: Example: GL_2

- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T, G) \cong S_2$.

Set $\mathcal{Z}(G, I) = Z(\mathcal{H}(G, I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G, I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

$$\begin{array}{ccccc} \mathcal{H}(G_r, K_r) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G_r, I_r) \\ \downarrow b_r & & \downarrow (\cdot)^r & & \downarrow b_r \\ \mathcal{H}(G, K) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G, I). \end{array}$$

Transition to Bernstein center: Example: GL_2

- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T,G) \cong S_2$.

Set $\mathcal{Z}(G,I) = Z(\mathcal{H}(G,I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G,I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

$$\begin{array}{ccccc} \mathcal{H}(G_r,K_r) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm,Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G_r,I_r) \\ \downarrow b_r & & \downarrow (\cdot)^r & & \downarrow b_r \\ \mathcal{H}(G,K) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm,Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G,I). \end{array}$$

Transition to Bernstein center: Example: GL_2

- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T, G) \cong S_2$.

Set $\mathcal{Z}(G, I) = Z(\mathcal{H}(G, I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G, I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

$$\begin{array}{ccccc} \mathcal{H}(G_r, K_r) & \xrightarrow[\sim]{\text{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\text{Bern}} & \mathcal{Z}(G_r, I_r) \\ \downarrow{\scriptstyle b_r} & & \downarrow{\scriptstyle (\cdot)^r} & & \downarrow{\scriptstyle b_r} \\ \mathcal{H}(G, K) & \xrightarrow[\sim]{\text{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\text{Bern}} & \mathcal{Z}(G, I). \end{array}$$

Transition to Bernstein center: Example: GL_2

- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T, G) \cong S_2$.

Set $\mathcal{Z}(G, I) = Z(\mathcal{H}(G, I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G, I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

$$\begin{array}{ccccc} \mathcal{H}(G_r, K_r) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G_r, I_r) \\ \downarrow b_r & & \downarrow (\cdot)^r & & \downarrow b_r \\ \mathcal{H}(G, K) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G, I). \end{array}$$

Transition to Bernstein center: Example: GL_2

- $I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{bmatrix} \supset I^+ = \begin{bmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{bmatrix}$.
- $T := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, and $W := W(T, G) \cong S_2$.

Set $\mathcal{Z}(G, I) = Z(\mathcal{H}(G, I))$, the center of the Iwahori-Hecke algebra.

$b_r : \mathcal{Z}(G_r, I_r) \to \mathcal{Z}(G, I)$ is defined analogously to the spherical case, but using the **Bernstein isomorphism**:

$$\begin{array}{ccccc} \mathcal{H}(G_r, K_r) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G_r, I_r) \\ \downarrow b_r & & \downarrow (\cdot)^r & & \downarrow b_r \\ \mathcal{H}(G, K) & \xrightarrow[\sim]{\mathrm{Sat}} & \mathbb{C}[X^\pm, Y^\pm]^{S_2} & \xleftarrow[\sim]{\mathrm{Bern}} & \mathcal{Z}(G, I). \end{array}$$

The Cartan decomposition gives a basis of spherical functions $f_\mu = 1_{K\mu(\varpi)K}$, (one for each dominant cocharacter $\mu \in X_*(T)$).

The Bernstein isomorphism gives a basis of Bernstein functions $z_\mu \in \mathcal{Z}(G, I)$ (one for each dominant $\mu \in X_*(T)$).

We have $b_r(z_\mu) = z_{r\mu}$.

(This is much simpler than the formula for $b_r(1_{K_r\mu(\varpi)K_r})$.)

The Cartan decomposition gives a basis of spherical functions $f_\mu = 1_{K\mu(\varpi)K}$, (one for each dominant cocharacter $\mu \in X_*(T)$).

The Bernstein isomorphism gives a basis of Bernstein functions $z_\mu \in \mathcal{Z}(G, I)$ (one for each dominant $\mu \in X_*(T)$).

We have $b_r(z_\mu) = z_{r\mu}$.

(This is much simpler than the formula for $b_r(1_{K_r\mu(\varpi)K_r})$.)

The Cartan decomposition gives a basis of spherical functions $f_\mu = 1_{K\mu(\varpi)K}$, (one for each dominant cocharacter $\mu \in X_*(T)$).

The Bernstein isomorphism gives a basis of Bernstein functions $z_\mu \in \mathcal{Z}(G, I)$ (one for each dominant $\mu \in X_*(T)$).

We have $b_r(z_\mu) = z_{r\mu}$.

(This is much simpler than the formula for $b_r(1_{K_r\mu(\varpi)K_r})$.)

The Cartan decomposition gives a basis of spherical functions $f_\mu = 1_{K\mu(\varpi)K}$, (one for each dominant cocharacter $\mu \in X_*(T)$).

The Bernstein isomorphism gives a basis of Bernstein functions $z_\mu \in \mathcal{Z}(G, I)$ (one for each dominant $\mu \in X_*(T)$).

We have $b_r(z_\mu) = z_{r\mu}$.

(This is much simpler than the formula for $b_r(1_{K_r\mu(\varpi)K_r})$.)

Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}^{G}_{N_r\delta}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

Walter Ray-Dulany (2010 PhD thesis) solved an analogous edge-counting problem in the tree, computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(z_\mu) = \mathrm{O}^{G}_{N_r\delta}(z_{r\mu}).$$

In general, direct computation is hopeless: orbital integrals are non-elementary.

Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}^{G}_{N_r\delta}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

Walter Ray-Dulany (2010 PhD thesis) solved an analogous edge-counting problem in the tree, computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(z_\mu) = \mathrm{O}^{G}_{N_r\delta}(z_{r\mu}).$$

In general, direct computation is hopeless: orbital integrals are non-elementary.

Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}^G_{N_r\delta}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

Walter Ray-Dulany (2010 PhD thesis) solved an analogous edge-counting problem in the tree, computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(z_\mu) = \mathrm{O}^G_{N_r\delta}(z_{r\mu}).$$

In general, direct computation is hopeless: orbital integrals are non-elementary.

Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}^G_{N_r\delta}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

Walter Ray-Dulany (2010 PhD thesis) solved an analogous edge-counting problem in the tree, computing both sides of

$$\mathrm{TO}^{G_r}_{\delta\theta}(z_\mu) = \mathrm{O}^G_{N_r\delta}(z_{r\mu}).$$

In general, direct computation is hopeless: orbital integrals are non-elementary.

Use of the tree

Langlands proved the spherical BCFL for GL_2 by computing both sides of

$$\mathrm{TO}_{\delta\theta}^{G_r}(1_{K_r\mu(\varpi)K_r}) = \mathrm{O}_{N_r\delta}^{G}(b_r(1_{K_r\mu(\varpi)K_r})),$$

for each dominant cocharacter $\mu \in X_*(T)$.

This was a vertex-counting problem on the tree for $\mathrm{SL}_2(F)$.

Walter Ray-Dulany (2010 PhD thesis) solved an analogous edge-counting problem in the tree, computing both sides of

$$\mathrm{TO}_{\delta\theta}^{G_r}(z_\mu) = \mathrm{O}_{N_r\delta}^{G}(z_{r\mu}).$$

In general, direct computation is hopeless: orbital integrals are non-elementary.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathfrak{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

Goal

Generalize the transfer homomorphism $b : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$ to Hecke algebras with deeper level, such that

1) it produces matching pairs, and

2) it is *spectrally explicit* (= defined geometrically on dual side).

We need the Bernstein center $\mathcal{Z}(G)$ of $G = G(F)$.

We will also need the local Langlands correspondence (LLC+)

We will show: LLC+ allows us to construct many matching pairs $f_r \leftrightarrow f^H$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.

The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.

The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\lim\limits_{\leftarrow} \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\lim\limits_{\leftarrow} \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X}$ = variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$.
Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\lim\limits_{\leftarrow} \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X} =$ variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting.
This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$.
Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X} =$ variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$.

Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.

The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X} =$ variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.

The Bernstein block $\mathcal{R}_\mathfrak{s}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

4 views of the Bernstein center

First step: construct many elements of the Bernstein center.
The $\mathbb{C}$-algebra $\mathcal{Z}(G)$ can be described as:

- ring of self natural-transformations of the identity functor.
- ess. compact G-invariant distributions on $\mathcal{H}(G)$.
- $\varprojlim \mathcal{Z}(G, J)$, where $J \subset G$ is compact-open s.g. and $\mathcal{Z}(G, J) = Z(\mathcal{H}(G, J))$.
- $\mathbb{C}[\mathfrak{X}]$ = regular functions on $\mathfrak{X} =$ variety of all s.c. supports $(M, \sigma)_G$.

Recall $\pi \in i_P^G(\sigma) \leftrightarrow (M, \sigma)_G = sc(\pi)$, and the $\mathbb{C}$-torus $X(M)$ of unramified characters on $M = M(F)$ acts on $\{(M, \sigma)_G\}_\sigma$ by twisting. This gives $\mathfrak{X}$ structure of a (disconnected) variety over $\mathbb{C}$. Connected components are the Bernstein varieties corresponding to the inertial classes $\mathfrak{s} = [M, \sigma]_G$.
The Bernstein block $\mathcal{R}_{\mathfrak{s}}(G)$ is the subcategory of representations whose s.c. support is of form $(M, \sigma\xi)_G$ for some $\xi \in X(M)$.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\text{Thus,} \quad \mathcal{Z}(G, I) = \mathbb{C}[\text{var. of } (T(F), \xi)_G : \xi \in X(T)]$$
$$= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W]$$
$$= \mathbb{C}[\widehat{T}(\mathbb{C})/W]$$
$$= \mathrm{Rep}(\widehat{G}).$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\begin{aligned} \text{Thus,} \quad \mathcal{Z}(G, I) &= \mathbb{C}[\text{var. of } (T(F), 1)_G, \ 1 \in X(T)] \\ &= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W] \\ &= \mathbb{C}[\widehat{T}(\mathbb{C})/W] \\ &= \mathrm{Rep}(\widehat{G}). \end{aligned}$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\text{Thus,} \quad \mathcal{Z}(G, I) = \mathbb{C}[\text{var. of } (T(F), \xi)_G, \ \xi \in X(T)]$$
$$= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W]$$

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\begin{aligned}\text{Thus, } \mathcal{Z}(G, I) &= \mathbb{C}[\text{var. of } (T(F), \xi)_G, \ \ \xi \in X(T)] \\ &= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W] \\ &= \mathbb{C}[\widehat{T}(\mathbb{C})/W] \\ &= \mathrm{Rep}(\widehat{G}).\end{aligned}$$

- $\mathcal{Z}(G, I) = \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\begin{aligned} \text{Thus,}\quad \mathcal{Z}(G, I) &= \mathbb{C}[\text{var. of } (T(F), \xi)_G,\ \ \xi \in X(T)] \\ &= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W] \\ &= \mathbb{C}[\widehat{T}(\mathbb{C})/W] \\ &= \mathrm{Rep}(\widehat{G}). \end{aligned}$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

Thus, $\mathcal{Z}(G, I) = \mathbb{C}[\text{var. of } (T(F), \xi)_G, \;\; \xi \in X(T)]$

$$= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W]$$
$$= \mathbb{C}[\widehat{T}(\mathbb{C})/W]$$
$$= \mathrm{Rep}(\widehat{G}).$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\text{Thus, } \mathcal{Z}(G, I) = \mathbb{C}[\text{var. of } (T(F), \xi)_G, \ \ \xi \in X(T)]$$
$$= \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W]$$
$$= \mathbb{C}[\widehat{T}(\mathbb{C})/W]$$
$$= \mathrm{Rep}(\widehat{G}).$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

Example: Iwahori block

- Assume I = Iwahori and T = Cartan (compatible with I...).
- Borel: $sc(\pi)$ is an unramified char.of $T(F)$ iff $\pi^I \neq 0$, and
- category of such G-reps is equivalent to $\mathcal{H}(G, I)$-Mod.

$$\text{Thus,}\ \ \mathcal{Z}(G, I) = \mathbb{C}[\text{var. of } (T(F), \xi)_G,\ \ \xi \in X(T)] \\ = \mathbb{C}[\mathrm{Hom}(T(F)/T(\mathcal{O}), \mathbb{C}^\times)/W] \\ = \mathbb{C}[\widehat{T}(\mathbb{C})/W] \\ = \mathrm{Rep}(\widehat{G}).$$

- $\mathcal{Z}(G, I) \cong \mathbb{C}[\widehat{T}/W]$ is the Bernstein isomorphism mentioned earlier.

To define a general "transfer homomorphism", we want to extend Satake/Bernstein to a natural map

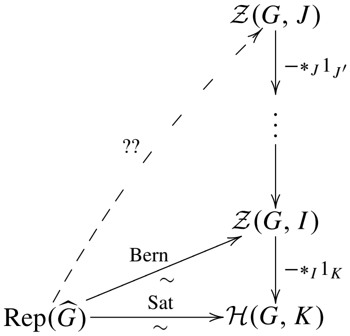
$$\mathrm{Rep}(\widehat{G}) \rightarrow \mathcal{Z}(G).$$

Can we complete the diagram?

That is, is there a canonical map, making the diagram commute?

$$\mathrm{Rep}(\widehat{G}) \to \mathcal{Z}(G)$$
$$V \mapsto Z_V.$$

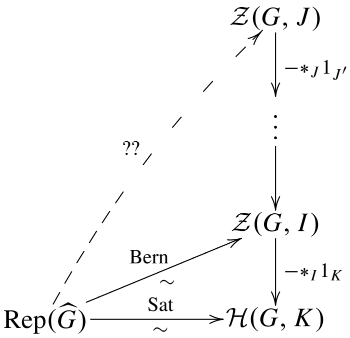
Can we complete the diagram?



That is, is there a canonical map, making the diagram commute?

$$\mathrm{Rep}(\widehat{G}) \to \mathcal{Z}(G)$$
$$V \mapsto Z_V.$$

Can we complete the diagram?



That is, is there a canonical map, making the diagram commute?

$$\mathrm{Rep}(\widehat{G}) \to \mathcal{Z}(G)$$
$$V \mapsto Z_V.$$

We can consider this for general unramified groups G (not just split groups), but then must replace $\widehat{G}$ with ${}^LG := \widehat{G} \rtimes W_F$, where W_F is the Weil group of F, and take $V \in \mathrm{Rep}({}^LG)$.

We will not consider ramified groups here.

We can consider this for general unramified groups G (not just split groups), but then must replace $\widehat{G}$ with ${}^LG := \widehat{G} \rtimes W_F$, where W_F is the Weil group of F, and take $V \in \mathrm{Rep}({}^LG)$.

We will not consider ramified groups here.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \;\mapsto\; Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \ \mapsto \ Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \mapsto Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \;\mapsto\; Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \ \mapsto \ Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Main construction

W_F Weil group of F, with inertia subgroup I_F, geometric Frob = Φ.
We need to assume LLC+, meaning for irred. $\pi \in \mathcal{R}(G)$:

1) existence of L-parameter $\varphi_\pi : W_F \to {}^L G = \widehat{G} \rtimes W_F$ with usual properties, and

2) Compatibility with $i_P^G(\cdot)$ (more below).

Proposition

G is arbitrary unramified and $V \in \mathrm{Rep}({}^L G)$. Assume LLC+ for G and its Levi subgroups. Then the function

$$\mathcal{R}_{\mathrm{irred}}(G) \ni \pi \ \mapsto \ Z_V(\pi) := \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V).$$

descends to give a regular function on $\mathfrak{X}$, thus an element $Z_V \in \mathcal{Z}(G)$.

Here $\mathrm{tr}^{ss}(\varphi_\pi(\Phi), V) := \mathrm{tr}(\varphi_\pi(\Phi), V^{\varphi_\pi(I_F)})$ is analogue of notion from ℓ-adic Galois representations ($\ell \neq p$).

In fact $Z_V \in \mathcal{Z}^{st}(G)$, the **stable** Bernstein center.

Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

$$\rho : \Gamma_F \to \mathrm{Aut}(V).$$

Grothendieck quasi-unipotence: $\exists$ finite-index subgroup of I_F acting purely unipotently on V.

Thus $\exists$ finite Γ_F-invariant filt. $F_\bullet(V)$ on V such that I_F acts through finite quotient on $gr := \oplus_i \mathrm{gr}^i(F_\bullet(V))$.

Definition

$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I)$.

Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

$$\rho : \Gamma_F \to \mathrm{Aut}(V).$$

Grothendieck quasi-unipotence: $\exists$ finite-index subgroup of I_F acting purely unipotently on V.

Thus $\exists$ finite Γ_F-invariant filt. $F_\bullet(V)$ on V such that I_F acts through finite quotient on $gr := \oplus_i \mathrm{gr}^i(F_\bullet(V))$.

Definition

$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I)$.

Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

$$\rho : \Gamma_F \to \mathrm{Aut}(V).$$

Grothendieck quasi-unipotence: $\exists$ finite-index subgroup of I_F acting purely unipotently on V.

Thus $\exists$ finite Γ_F-invariant filt. $F_\bullet(V)$ on V such that I_F acts through finite quotient on $gr := \oplus_i \mathrm{gr}^i(F_\bullet(V))$.

Definition

$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I)$.

Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

$$\rho : \Gamma_F \to \mathrm{Aut}(V).$$

Grothendieck quasi-unipotence: ∃ finite-index subgroup of I_F acting purely unipotently on V.

Thus ∃ finite Γ_F-invariant filt. $F_\bullet(V)$ on V such that I_F acts through finite quotient on $gr := \oplus_i \mathrm{gr}^i(F_\bullet(V))$.

Definition

$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I).$

Aside: semisimple trace

Notion is due to Rapoport.

Fix $\ell \neq p = \mathrm{char}(\mathcal{O}/\varpi\mathcal{O})$. Let V be a finite-dimensional $\bar{\mathbb{Q}}_\ell$-space with a continuous representation

$$\rho : \Gamma_F \to \mathrm{Aut}(V).$$

Grothendieck quasi-unipotence: $\exists$ finite-index subgroup of I_F acting purely unipotently on V.

Thus $\exists$ finite Γ_F-invariant filt. $F_\bullet(V)$ on V such that I_F acts through finite quotient on $gr := \oplus_i \mathrm{gr}^i(F_\bullet(V))$.

Definition

$\mathrm{tr}^{ss}(\Phi, V) = \mathrm{tr}(\Phi, gr^I)$.

Aside: semisimple L-functions

To

$$W_{\mathbb{Q}_p} \xrightarrow{\varphi_\pi} {}^L G \xrightarrow{r_V} \mathrm{Aut}(V)$$

we associate

Definition

$$\log(L^{ss}(s, \pi, r_V)) = \sum_{r \geq 1} \mathrm{tr}^{ss}(\varphi_\pi(\Phi^r), \; V) \frac{p^{-rs}}{r}.$$

One reason why: can express $Z_{\mathfrak{p}}^{ss}(s, Sh)$ in terms of several $L^{ss}(s-?, \pi_p, r)$, where r is determined by Sh.

Aside: semisimple L-functions

To

$$W_{\mathbb{Q}_p} \xrightarrow{\varphi_\pi} {}^L G \xrightarrow{r_V} \mathrm{Aut}(V)$$

we associate

Definition

$\log(L^{ss}(s, \pi, r_V)) = \sum_{r \geq 1} \mathrm{tr}^{ss}(\varphi_\pi(\Phi^r), V) \frac{p^{-rs}}{r}$.

One reason why: can express $Z^{ss}_{\mathfrak{p}}(s, Sh)$ in terms of several $L^{ss}(s-?, \pi_p, r)$, where r is determined by Sh.

Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \to {}^L G$ and $\varphi_\sigma : W_F \to {}^L M \hookrightarrow {}^L G$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

(b) G unramified, V arbitrary, and $J = I, I^+$, or parahoric.

Hope: LLC+ for groups such as GSp_{2n} will come eventually from an extension of Arthur's forthcoming book.

Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \to {}^LG$ and $\varphi_\sigma : W_F \to {}^LM \hookrightarrow {}^LG$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

(b) G unramified, V arbitrary, and $J = I, I^+$, or parahoric.

Hope: LLC+ for groups such as GSp_{2n} will come eventually from an extension of Arthur's forthcoming book.

Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \rightarrow {}^L G$ and $\varphi_\sigma : W_F \rightarrow {}^L M \hookrightarrow {}^L G$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

(b) G unramified, V arbitrary, and $J = I, I^+$, or parahoric.

Hope: LLC+ for groups such as GSp_{2n} will come eventually from an extension of Arthur's forthcoming book.

Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \to {}^L G$ and $\varphi_\sigma : W_F \to {}^L M \hookrightarrow {}^L G$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

(b) G unramified, V arbitrary, and $J = I, I^+$, or parahoric.

Hope: LLC+ for groups such as GSp_{2n} will come eventually from an extension of Arthur's forthcoming book.

Back to $Z_V(\pi) = \mathrm{tr}^{ss}(\varphi_\pi(\Phi), V)$

Construction unconditional in cases:

Compatibility with $i_P^G(\cdot)$: if $\pi \in i_P^G(\sigma)$, then $\varphi_\pi : W_F \to {}^L G$ and $\varphi_\sigma : W_F \to {}^L M \hookrightarrow {}^L G$ are $\widehat{G}$-conjugate.

OK for GL_n (Bernstein-Zelevinsky + Jacquet)

Can define Z_V without LLC+ at least in cases:

(a) $V = \mathbb{C}^n$ is std. rep. of $\mathrm{GL}_n(\mathbb{C})$ (Scholze:LLC)

(but for general $V \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ we need LLC for GL_n).

(b) G unramified, V arbitrary, and $J = I, I^+$, or parahoric.

Hope: LLC+ for groups such as GSp_{2n} will come eventually from an extension of Arthur's forthcoming book.

Scholze's observation

If $\pi \in \mathcal{R}_{\mathrm{irred}}(\mathrm{GL}_n(\mathbb{Q}_p))$ is a subquotient of normalized induction of $\pi_1 \boxtimes \cdots \boxtimes \pi_t \in \mathcal{R}(\mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_t}(\mathbb{Q}_p))$, then

$$\mathrm{tr}^{ss}(\varphi_\pi(\Phi), \mathbb{C}^n) = \sum_{\pi_i} \pi_i(p^r)$$

the sum taken over π_i which are **unramified characters.**

One can simply work with the RHS in place of invoking LLC+ for GL_n.

Scholze's observation

If $\pi \in \mathcal{R}_{\mathrm{irred}}(\mathrm{GL}_n(\mathbb{Q}_p))$ is a subquotient of normalized induction of $\pi_1 \boxtimes \cdots \boxtimes \pi_t \in \mathcal{R}(\mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_t}(\mathbb{Q}_p))$, then

$$\mathrm{tr}^{ss}(\varphi_\pi(\Phi), \mathbb{C}^n) = \sum_{\pi_i} \pi_i(p^r)$$

the sum taken over π_i which are **unramified characters.**

One can simply work with the RHS in place of invoking LLC+ for GL_n.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.
Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.

Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z^H_V := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z^H_V$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z^H_V * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.
Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.

Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.

Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.

Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.

Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z^H_V := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z^H_V$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z^H_V * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.
Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z_V^H := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z_V^H$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z_V^H * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.

Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

The Ztransfer Conjecture (We explain only the case $r = 1$: standard endoscopy.)

Suppose given $(H, s, {}^L\eta)$, ${}^L\eta : {}^LH \to {}^LG$. We may restrict $V \in \mathrm{Rep}({}^LG)$ to $V|{}^L\eta \in \mathrm{Rep}({}^LH)$. Get $Z^H_V := Z_{V|{}^L\eta} \in \mathcal{Z}^{st}(H)$.

Conjecture

For every $V \in \mathrm{Rep}({}^LG)$, we have $Z_V \leftrightarrow Z^H_V$ in the sense of distributions:

$$f \leftrightarrow f^H \Longrightarrow Z_V * f \leftrightarrow Z^H_V * f^H.$$

There exists similar conjecture in general Frobenius twisted case.
Inputting FL $1_K \leftrightarrow 1_{K_H}$ yields Example C: Hales spherical transfer.
Inputting $1_{K_r} \leftrightarrow 1_K$ (Kottwitz units) yields Example A: Clozel-Labesse BCFL.
Inputting $1_{K_r} \leftrightarrow 1_{K_H}$ (Waldspurger) yields simultaneous generalization of the above and of Morel's Frobenius twisted transfer theorem for spherical Hecke algebras of certain classical groups.

Aside: explicit matching pairs

With the FL and Waldspurger's transfer now available, it makes sense to try to find explicit matching pairs $f_r \leftrightarrow f^H$ where f belongs to a prescribed family.

Example: Kazhdan-Varshavsky: f and f^H are Deligne-Lusztig functions coming from the representation theory of finite groups of Lie type.

The Ztransfer conjecture would add many examples to this general program.

Aside: explicit matching pairs

With the FL and Waldspurger's transfer now available, it makes sense to try to find explicit matching pairs $f_r \leftrightarrow f^H$ where f belongs to a prescribed family.

Example: Kazhdan-Varshavsky: f and f^H are Deligne-Lusztig functions coming from the representation theory of finite groups of Lie type.

The Ztransfer conjecture would add many examples to this general program.

Aside: explicit matching pairs

With the FL and Waldspurger's transfer now available, it makes sense to try to find explicit matching pairs $f_r \leftrightarrow f^H$ where f belongs to a prescribed family.

Example: Kazhdan-Varshavsky: f and f^H are Deligne-Lusztig functions coming from the representation theory of finite groups of Lie type.

The Ztransfer conjecture would add many examples to this general program.

A heuristic proof of the Ztransfer Conjecture can be given, assuming enough expected properties about L-packets.

Question: What can be proved unconditionally?

A heuristic proof of the Ztransfer Conjecture can be given, assuming enough expected properties about L-packets.

Question: What can be proved unconditionally?

Results: Base change fundamental lemmas – essentially Iwahori level

Consider $H = G$, i.e., $f_r \leftrightarrow f$ has the sense of stable base change.

Theorem (H. 2009, 2010 – Predecessor of Ztransfer conjecture)

For G unramified over F, and $J = I, I^+$, or parahoric, there exists a base-change homomorphism

$$b_r : \mathcal{Z}(G_r, J_r) \to \mathcal{Z}(G, J)$$

defined explicitly in terms of the Bernstein isomorphism (on the dual side) with the property

$$f_r \leftrightarrow b_r(f_r).$$

This was used to study certain Shimura varieties with parahoric or $\Gamma_1(p)$-level structure at p (see below).

Results: Base change fundamental lemmas – essentially Iwahori level

Consider $H = G$, i.e., $f_r \leftrightarrow f$ has the sense of stable base change.

Theorem (H. 2009, 2010 – Predecessor of Ztransfer conjecture)

For G unramified over F, and $J = I, I^+$, or parahoric, there exists a base-change homomorphism

$$b_r : \mathcal{Z}(G_r, J_r) \to \mathcal{Z}(G, J)$$

defined explicitly in terms of the Bernstein isomorphism (on the dual side) with the property

$$f_r \leftrightarrow b_r(f_r).$$

This was used to study certain Shimura varieties with parahoric or $\Gamma_1(p)$-level structure at p (see below).

Results: Base change fundamental lemmas – essentially Iwahori level

Consider $H = G$, i.e., $f_r \leftrightarrow f$ has the sense of stable base change.

Theorem (H. 2009, 2010 – Predecessor of Ztransfer conjecture)

For G unramified over F, and $J = I, I^+$, or parahoric, there exists a base-change homomorphism

$$b_r : \mathcal{Z}(G_r, J_r) \to \mathcal{Z}(G, J)$$

defined explicitly in terms of the Bernstein isomorphism (on the dual side) with the property

$$f_r \leftrightarrow b_r(f_r).$$

This was used to study certain Shimura varieties with parahoric or $\Gamma_1(p)$-level structure at p (see below).

Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow 1_{J_H}$.

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow 1_{J_H}$.

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow 1_{J_H}$.

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow 1_{J_H}$.

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

Results: GL_n – arbitrary level

Proposition (H., Scholze)

The (Frobenius-twisted) Ztransfer Conjecture holds for $G = \mathrm{GL}_n$.

(Congruence-level FL: Ferrari 2007) If $J = K(N) \subset G(F)$ any *principal congruence subgroup* for a *split* group G, then $1_{J_r} \leftrightarrow 1_{J_H}$.

Therefore, we get many pairs of matching functions in centers of principal-congruence-group-level Hecke algebras for GL_n (we do not know the structure of these centers!).

For more general groups, one must work harder and impose restrictions on the level.

Theorem (H. in preparation)

If G split with connected center, then the Ztransfer conjecture holds with $J = I$ or I^+.

Will be applied to GSp_{2n}-Shimura varieties with $\Gamma_1(p)$-level at p.

For more general groups, one must work harder and impose restrictions on the level.

Theorem (H. in preparation)

If G split with connected center, then the Ztransfer conjecture holds with $J = I$ or I^+.

Will be applied to GSp_{2n}-Shimura varieties with $\Gamma_1(p)$-level at p.

Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^pK_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^LG = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set
  $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: “clean form” in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_{\mathfrak{p}}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. “fake unitary case” with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^LG = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: “clean form” in a special case

- $Sh = Sh(\mathbf{G}, X, K^p K_p)$, PEL over $\mathcal{O}_{E_\mathfrak{p}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. “fake unitary case” with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture: "clean form" in a special case

- $Sh = Sh(\mathbf{G}, X, K^pK_p)$, PEL over $\mathcal{O}_{E_{\mathfrak{p}}}$ $(= \mathbb{Z}_p)$ (for simplicity), and $G = \mathbf{G}_{\mathbb{Q}_p}$ *split*.
- no endoscopy, e.g. "fake unitary case" with $\mathbf{G}(\mathbb{R}) \cong \mathrm{GU}(d, n-d)$.
- Let $\mu := \mu_h \in X_*(T) = X^*(\widehat{T})$ the **Shimura cocharacter**, with dual cocharacter μ^*.
- Let $V = V_{\mu^*}$ denote the representation of ${}^LG = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ with extreme weight μ^*.
- Set $f_{r,1} = p^{\frac{1}{2}\dim(Sh)} \times$ image of base-change operation applied to Z_V.

Test function conjecture – clean form

Conjecture (H.-Kottwitz – "clean form")

In the situation above, for every $r \geq 1$, the test function $f_{r,1}$ above satisfies: the alternating sum of the semi-simple traces

$$\sum_{i=0}^{2\dim(Sh)} (-1)^i \operatorname{tr}^{ss}(\Phi_{\mathfrak{p}}^r, \mathrm{H}^i(Sh \times_E \bar{E}_{\mathfrak{p}}, \bar{\mathbb{Q}}_\ell))$$

equals the trace

$$\operatorname{tr}(1_{K^p} \otimes f_{r,1} \otimes f_\infty, \mathrm{L}^2(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^\circ \backslash \mathbf{G}(\mathbb{A}))).$$

In fact, this is really a consequence of the "real" Test Function Conjecture together with the Ztransfer conjecture.

Test function conjecture – clean form

Conjecture (H.-Kottwitz – “clean form”)

In the situation above, for every $r \geq 1$, the test function $f_{r,1}$ above satisfies: the alternating sum of the semi-simple traces

$$\sum_{i=0}^{2\dim(Sh)} (-1)^i \mathrm{tr}^{ss}(\Phi_{\mathfrak{p}}^r, \mathrm{H}^i(Sh \times_E \bar{E}_{\mathfrak{p}}, \bar{\mathbb{Q}}_\ell))$$

equals the trace

$$\mathrm{tr}(1_{K^p} \otimes f_{r,1} \otimes f_\infty, \mathrm{L}^2(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^\circ \backslash \mathbf{G}(\mathbb{A}))).$$

In fact, this is really a consequence of the “real” Test Function Conjecture together with the Ztransfer conjecture.

Consequence: Automorphy of local factors of Hasse-Weil Zeta functions

Remark

When the above equation holds (e.g. in certain "nice" cases of "unitary" Shimura varieties as above when both Ztransfer conjecture and "real" test function conjecture are known), we have

$$Z^{ss}_{\mathfrak{p}}(s, Sh) = \prod_{\pi_f} L^{ss}\left(s - \frac{\dim Sh}{2}, \pi_p, r_{\mu^*}\right)^{n(\pi_f)} \tag{1}$$

where $\pi_f = \pi^{p,\infty} \otimes \pi_p$ *ranges over certain representations of* $\mathbf{G}(\mathbb{A}_f)$ *and* $n(\pi_f) \in \mathbb{Z}$.

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ is split of type A or C and K_p is parahoric.

Theorem (H.-Rapoport, 2010)

A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$), if $K_p = I^+$.

Theorem (Scholze, 2010)

It holds in the Harris-Taylor case ($\mathrm{GU}(1, n-1)$ and Sh proper), if K_p is any congruence subgroup.

Currently working on $\mathrm{GSp}(2n)$-cases, with $\Gamma_1(p)$-level...

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ is split of type A or C and K_p is parahoric.

Theorem (H.-Rapoport, 2010)

A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$), if $K_p = I^+$.

Theorem (Scholze, 2010)

It holds in the Harris-Taylor case ($\mathrm{GU}(1, n-1)$ and Sh proper), if K_p is any congruence subgroup.

Currently working on $\mathrm{GSp}(2n)$-cases, with $\Gamma_1(p)$-level...

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ is split of type A or C and K_p is parahoric.

Theorem (H.-Rapoport, 2010)

A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$), if $K_p = I^+$.

Theorem (Scholze, 2010)

It holds in the Harris-Taylor case ($\mathrm{GU}(1, n-1)$ and Sh proper), if K_p is any congruence subgroup.

Currently working on $\mathrm{GSp}(2n)$-cases, with $\Gamma_1(p)$-level...

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ *is split of type* A *or* C *and* K_p *is parahoric.*

Theorem (H.-Rapoport, 2010)

A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$)*, if* $K_p = I^+$.

Theorem (Scholze, 2010)

It holds in the Harris-Taylor case ($\mathrm{GU}(1, n-1)$ *and* Sh *proper), if* K_p *is any congruence subgroup.*

Currently working on $\mathrm{GSp}(2n)$-cases, with $\Gamma_1(p)$-level...

The test function conjecture has been proved in some cases

Theorem (H.-Ngô, 2002, 2005, 2009)

The (real) test function conjecture holds if $\mathbf{G}_{\mathbb{Q}_p}$ is split of type A or C and K_p is parahoric.

Theorem (H.-Rapoport, 2010)

A stronger version holds in the Drinfeld case ($\mathrm{GU}(1, n-1)$), if $K_p = I^+$.

Theorem (Scholze, 2010)

It holds in the Harris-Taylor case ($\mathrm{GU}(1, n-1)$ and Sh proper), if K_p is any congruence subgroup.

Currently working on $\mathrm{GSp}(2n)$-cases, with $\Gamma_1(p)$-level...

Summary

The usual statements of the FL and transfer homomorphisms fit into a larger framework, defined on part of the Bernstein center in an explicit way.

In certain cases, the construction is unconditional, and sometimes can be proved.

For GSp_{2n}, there is hope it can be established and used to study the associated non-compact Shimura varieties with bad reduction.

THE END.

Summary

The usual statements of the FL and transfer homomorphisms fit into a larger framework, defined on part of the Bernstein center in an explicit way.

In certain cases, the construction is unconditional, and sometimes can be proved.

For GSp_{2n}, there is hope it can be established and used to study the associated non-compact Shimura varieties with bad reduction.

THE END.

Summary

The usual statements of the FL and transfer homomorphisms fit into a larger framework, defined on part of the Bernstein center in an explicit way.

In certain cases, the construction is unconditional, and sometimes can be proved.

For GSp_{2n}, there is hope it can be established and used to study the associated non-compact Shimura varieties with bad reduction.

THE END.

Summary

The usual statements of the FL and transfer homomorphisms fit into a larger framework, defined on part of the Bernstein center in an explicit way.

In certain cases, the construction is unconditional, and sometimes can be proved.

For GSp_{2n}, there is hope it can be established and used to study the associated non-compact Shimura varieties with bad reduction.

THE END.