T. J. Haines (2017) "Correction to "On Satake parameters for representations with parahoric fixed vectors","
International Mathematics Research Notices, Vol. 2017, No. 00, pp. 1–11
doi: 10.1093/imrn/rnx088

Correction to "On Satake parameters for representations with parahoric fixed vectors"

Thomas J. Haines*

Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA

Correspondence to be sent to: e-mail: tjh@math.umd.edu

The main purpose of this note is to correct the statement of [5, Lemma 8.2]. The source of the mistake was an erroneous result in a previous article, namely [4, Lemma 11.12.3]. This note proves corrected versions of both lemmas, and provides counterexamples to the original statements.

1 Correction of a Lemma in [4]

The published statement of [4, Lemma 11.12.3] is false. It should be replaced by the following statement. We use the notation as in [4, Section 11] throughout this note, unless otherwise stated.

Lemma 11.12.3′. Recall that $T^* = \mathrm{Cent}_{G^*}(A^*)$ is a maximal torus in G^* defined over F; let S^* be the F^{un}-split component of T^*, a maximal F^{un}-split torus in G^* defined over F and containing A^*. We have $T^* = \mathrm{Cent}_{G^*}(A^*) = \mathrm{Cent}_{G^*}(S^*)$. Choose a maximal F^{un}-split torus $S \subset G$ which is defined over F and which contains A, and set $T = \mathrm{Cent}_G(S)$. Choose $\psi_0 \in \Psi_M$ such that ψ_0 is defined over F^{un} and satisfies $\psi_0(S) = S^*$ and hence $\psi_0(T) = T^*$.

Received March 6, 2017; Revised March 6, 2017; Accepted April 3, 2017

Then the diagram

$$\begin{tikzcd} W(G,A) \arrow[r, hook, "\psi_0^\natural"] \arrow[d, "\wr"'] & W(G^*,A^*)/W(M^*,A^*) \arrow[d, "\wr"'] \\ \left[W(G,S)/W(M,S)\right]^{\Phi_F} \arrow[r, hook, "\psi_0"] & \left[W(G^*,S^*)/W(M^*,S^*)\right]^{\Phi_F^*} \end{tikzcd} \tag{1.1}$$

defines an injective map $\psi_0^\natural$. It depends on the choice of the data P, B^* used to define Ψ_M and M^*, but it is independent of the choices of S and $\psi_0 \in \Psi_M$ with the stated properties. □

Proof. The left vertical arrow is [6, Lemma 6.1.2]. The right vertical arrow is described in [6, Proposition 12.1.1]. The proof of the latter justifies the lower horizontal arrow. Indeed, given $w \in W(G,A)$ we may choose a representative $n \in N_G(S)(L)^{\Phi_F}$ (cf. [6]). We have $\psi_0^{-1} \circ \Phi_F^* \circ \psi_0 \circ \Phi_F^{-1} = \mathrm{Int}(m_\Phi)$ for some $m_\Phi \in N_M(S)(L)$. Since n is Φ_F-fixed, we get

$$\Phi_F^*(\psi_0(n)) = \psi_0(n) \cdot \left[\psi_0(n)^{-1}\psi_0(m_\Phi)\psi_0(n)\psi_0(m_\Phi)^{-1}\right].$$

As n normalizes M and hence $\psi_0(n)$ normalizes M^*, this shows that $\psi_0(n)W(M^*,S^*)$ is Φ_F^*-fixed.

There exists $m_n^* \in N_{M^*}(S^*)(L)$ such that $\psi_0(n)m_n^* \in N_{G^*}(A^*)(F)$. Then $\psi_0^\natural(w)$ is the image of $\psi_0(n)m_n^*$ in $W(G^*,A^*)/W(M^*,A^*)$. The independence statement is proved using this description. ■

Remark 1.1. The original lemma asserted that ψ_0 and $\psi_0^\natural$ are *bijective*. This usually fails when G is not isomorphic to G^*: for example, take $G = \mathrm{GL}_2(D)$, where D is a central quaternion algebra over a p-adic field F; then $W(G,A) \cong S_2$, whereas $W(G^*,A^*)/W(M^*,A^*) \cong S_4/(S_2 \times S_2)$. □

Remark 1.2. The proof above shows that the obvious isomorphism

$$\psi_0 : W(G,S)/W(M,S) \xrightarrow{\sim} W(G^*,S^*)/W(M^*,S^*)$$

maps Φ-invariant cosets to Φ^*-invariant cosets. The reason the converse direction does not hold is that an element $n^* \in N_{G^*}(S^*)(L)^{\Phi^*}$ need not normalize M^*. Clearly, the image of the second horizontal arrow in (1.1) consists of those cosets which are invariant under the transported action of Φ, namely under $\psi_0 \circ \Phi \circ \psi_0^{-1} =: \Phi_{\psi_0}$. More usefully for our present purposes, an element $n^* \in N_{G^*}(S^*)(L)^{\Phi^*}$ represents the image of a Φ-fixed coset if n^* normalizes M^*. $\square$

1.1 Impact on other results of [4]

The mistake in Lemma 11.12.3 had no impacts on other results of [4]. The subsequent result, [4, Lemma 11.12.4] proves the existence of the *normalized transfer homomorphisms*

$$\mathbb{C}[T^*(F)/T^*(F)_1]^{W(G^*,A^*)} \xrightarrow{\tilde{t}_{A^*,A}} \mathbb{C}[M(F)/M(F)_1]^{W(G,A)}$$

which play a key role in [4, 5]. Fortunately, the proof of [4, Lemma 11.12.4] goes through as written. Its proof used the construction of n and m_n^* given in the proof of Lemma 11.12.3, but not the false surjectivity assertion of that lemma.

2 Correction of a lemma in [5]

2.1 Statement

The goal of this section is to give the corrected statement of [5, Lemma 8.2]. We will use the notation $\tilde{\psi}_0$ for the map denoted by $\tilde{t}_{A^*,A}$ in [5, (8.1)]

$$\begin{aligned} \tilde{\psi}_0 : Z(\widehat{M}^I)_\Phi &\longrightarrow (\widehat{T^*}^{I^*})_{\Phi^*} \\ \hat{m} &\longmapsto \delta_{B^*}^{-1/2} \cdot \hat{\psi}_0(\delta_P^{1/2}\hat{m}) \end{aligned}$$

and $\bar{\psi}_0$ for the map $Z(\widehat{M}^I)_\Phi/W(G,A) \to (\widehat{T^*}^{I^*})_{\Phi^*}/W(G^*,A^*)$ which it induces. The original statement of [5, Lemma 8.2] was that $\bar{\psi}_0$ is a closed immersion. We shall show in Sections 2.4 and 2.5 that this is indeed the case for $G^* = \mathrm{GL}_n$ but is false in general.

Lemma 8.2´. The morphism

$$\bar{\psi}_0 \;:\; (Z(\widehat{M})^I)_\Phi/W(G,A) \;\to\; (\widehat{T^*}^{I^*})_{\Phi^*}/W(G^*,A^*) \tag{2.1}$$

is a finite morphism which is birational on to its image. $\square$

Note that $\bar{\psi}_0$ is an isomorphism when G is quasi-split, so this issue only arises for non-quasisplit groups.

2.2 Preliminaries on dual groups

Lemma 2.1. Suppose M is an F-Levi subgroup of G, hence is a Levi factor of a parabolic subgroup $P = MN$ defined over F. Then there exists an embedding $\widehat{M} \subset \widehat{G}$ such that the Γ-action on $\widehat{M}$ is inherited from the Γ-action on $\widehat{G}$. □

Proof. We follow the dual group conventions of Kottwitz [7, Section 1]. Since $M \subset P$ are Γ-stable subgroups of G, we may define the non-rigidified dual group $\widehat{G}$ together with its Γ-action by taking it to be the group corresponding to the based root system which is the inverse-limit over all pairs (B, T) with $T \subset M$ and $B \subset P$ of the corresponding based root systems

$$\operatorname*{proj\,lim}_{(B,T)} (X_*(T), \Delta^\vee(B,T), X^*(T), \Delta(B,T)).$$

The Galois action permutes the pairs (B, T) where $T \subset M$ and $B \subset P$, and we may use conjugation by an element of P to move back to the original pair. We have $B = B_M \cdot N$, and the Galois action and the conjugation action both leave N fixed and move only the Borels $B_M = B \cap M$ around. We may therefore use only conjugation by M to describe the Galois action on such pairs (B, T). The based root system

$$\operatorname*{proj\,lim}_{(B_M,T)} (X_*(T), \Delta^\vee(B_M,T), X^*(T), \Delta(B_M,T))$$

is used to define $\widehat{M}$. We can now endow both of the above based root systems with splittings, which are compatible in the obvious sense that the one for $\widehat{M}$ appears inside the one for $\widehat{G}$. The above inverse limits endowed with these splittings pin down $\widehat{M}$ and $\widehat{G}$ uniquely, in a way compatible with the Γ-actions. ■

Lemma 2.2. Let $M \subset G$ be an F-Levi subgroup. Fix an embedding $\widehat{M} \subset \widehat{G}$ as in Lemma 2.1. Let I denote the inertial subgroup of $\Gamma = \mathrm{Gal}(\bar{F}/F)$ (similar results hold with Γ in place of I).

(a) The group $\widehat{M}^{I,\circ}$ is a Levi subgroup of the reductive group $\widehat{G}^{I,\circ}$.
(b) $\widehat{M}^I = Z(\widehat{G})^I \cdot \widehat{M}^{I,\circ}$.
(c) $Z(\widehat{M}^I) = Z(\widehat{G})^I \cdot Z(\widehat{M}^{I,\circ})$.
(d) $Z(\widehat{M}^{I,\circ}) \subset Z(\widehat{M})^I$.
(e) $Z(\widehat{M}^I) = Z(\widehat{M})^I$. □

Proof. We fix a maximal F-torus $T \subset M$.

Part (a): We may assume the maximal torus $\widehat{T}^{I,\circ} \subset \widehat{G}^{I,\circ}$ is contained in $\widehat{M}^{I,\circ}$. Then the latter is generated by $\widehat{T}^{I,\circ}$ and the root groups $U_{\bar{\alpha}}$ which are contained in $\widehat{M}^{I,\circ}$. Here $\bar{\alpha}$ ranges over certain roots of $\Phi(\widehat{G}^{I,\circ}, \widehat{T}^{I,\circ}) = [(\Phi^\vee)^\diamond]_{\mathrm{red}}$ (cf. [3, Proposition 5.1]); these are just the *shorter* of the restrictions of roots in $\Phi^\vee = \Phi(\widehat{G}, \widehat{T})$ to $\widehat{T}^{I,\circ}$. Those roots $\bar{\alpha}$ such that $U_{\bar{\alpha}} \subset \widehat{M}^{I,\circ}$ must be restrictions from $\Phi(\widehat{M}, \widehat{T})$, since $U_{\bar{\alpha}}$ is contained in the subgroup of $\widehat{G}$ generated by $U_{\tau(\alpha)}$ for $\tau \in I$. One can then check that the root system for $(\widehat{M}^{I,\circ}, \widehat{T}^{I,\circ})$ is the root system of a Levi subgroup of $\widehat{G}^{I,\circ}$ (use the criterion [2, Proposition 1.20(ii)] – take any point $x \in X_*(\widehat{T})_{\mathbb{R}}$ showing $\widehat{M}$ is a Levi subgroup and replace it by its I-average). Thus $\widehat{M}^{I,\circ}$ is a Levi subgroup.

Part (b): Recall that $\widehat{G}^I = Z(\widehat{G})^I \cdot \widehat{G}^{I,\circ}$ and $\pi_0(\widehat{T}^I) = \pi_0(\widehat{G}^I)$ ([5, Proposition 4.1, Section 5]). Applying these for $\widehat{M} = \widehat{G}$ as well we get $Z(\widehat{G})^I$ surjects on to $\pi_0(\widehat{T}^I) = \pi_0(\widehat{M}^I)$, proving (b). (It is legitimate to apply these results to the Galois action inherited on $\widehat{M}$ by Lemma 2.1.) Part (c) follows immediately.

Parts (d,e): Using part (a), clearly $Z(\widehat{M})^{I,\circ}$ and $Z(\widehat{M}^{I,\circ})$ both belong to $\widehat{T}^{I,\circ}$. Let $t \in Z(\widehat{M}^{I,\circ})$. Then for every root $\bar{\alpha} \in \Phi(\widehat{M}^{I,\circ}, \widehat{T}^{I,\circ})$ (the restriction of some root $\alpha \in \Phi(\widehat{M}, \widehat{T})$), we have

$$\alpha(t) = \bar{\alpha}(t) = 1$$

the latter equality since the center of a Levi subgroup is the kernel of all roots for that subgroup. This equality holds for the *shorter* restrictions, and therefore for all restrictions, that is, for all α. This shows that $t \in Z(\widehat{M})$ for the same reason as before. This proves $Z(\widehat{M}^{I,\circ}) \subset Z(\widehat{M})^I$, that is, part (d). Together with $Z(\widehat{M}^I) = Z(\widehat{M})^I \cdot Z(\widehat{M}^{I,\circ})$ (which follows from (c)), we get (e) as well. ■

For convenience, we write the simple roots $\Delta(\widehat{M^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ}) = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\}$ and

$$\Delta(\widehat{G^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ}) \backslash \Delta(\widehat{M^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ}) = \{\bar{\beta}_1, \ldots, \bar{\beta}_k\}.$$

Let f be the order of Φ^* acting on $\widehat{G^*}$. Fix an integer $r > \max\{\mathrm{ht}(\alpha) \mid \alpha \in \Phi(\widehat{G^*}, \widehat{T^*})\}$. Recall q is the order of the residue field of F.

Definition 2.3. We construct the following sets:

(a) Let $\mathfrak{T}^* \subset Z(\widehat{M^*}^{\Gamma^*})$ be the open dense subset whose elements z^* satisfy: given any integers $d_1, \ldots, d_k$ with $0 \leq d_i \leq r$, $\forall i$ and $d_1 + \cdots + d_k > 0$, we have

$$(\bar{\beta}_1(z^*))^{d_1} \cdots (\bar{\beta}_k(z^*))^{d_k} \neq q^{fj}$$

for all integers j with $|j| \leq r$ (we use Lemma 2.2(c) to prove density).

(b) Let $\mathfrak{T}_* \subset (Z(\widehat{M^*})^{I^*})_{\Phi^*}$ be the inverse image of $\mathfrak{T}^*$ under the norm map $N : (Z(\widehat{M^*})^{I^*})_{\Phi^*} \to Z(\widehat{M^*}^{\Gamma^*})$; this is open and dense since N is an isogeny. □

For $z_1, z_2 \in (Z(\widehat{M})^I)_\Phi$, write $z_i^* := \hat{\psi}_0(z_i)$ for their images in $(Z(\widehat{M^*})^{I^*})_{\Phi^*}$. Then $\tilde{\psi}_0(z_i) = \delta_{B_{M^*}}^{-1/2} z_i^* \in \delta_{B_{M^*}}^{-1/2}(Z(\widehat{M^*})^{I^*})_{\Phi^*}$.

Lemma 2.4. Suppose z_1^* and z_2^* belong to $\mathfrak{T}_*$ and are such that $\delta_{B_{M^*}}^{-1/2} z_1^*$ and $\delta_{B_{M^*}}^{-1/2} z_2^*$ are $W(G^*, A^*)$-conjugate. Then z_1 and z_2 are $W(G, A)$-conjugate. □

Proof. Using the identifications $W(G^*, A^*) = W(\widehat{G^*}, \widehat{T^*})^{\Gamma^*} = W(\widehat{G^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ})$ (see [5, Propositions 4.1(c) and 6.1]), we have a relation of the form

$$\delta_{B_{M^*}}^{-1/2}\, z_1^* = n^*(\delta_{B_{M^*}}^{-1/2}\, z_2^*)n^{*-1} = {}^{w^*}(\delta_{B_{M^*}}^{-1/2})\, n^* z_2^* n^{*-1} \tag{2.2}$$

for an element $n^* \in N_{\widehat{G^*}^{\Gamma^*,\circ}}(\widehat{T^*}^{\Gamma^*,\circ})$ representing an element $w^* \in W(\widehat{G^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ})$. Note that $\delta_{B_{M^*}}^{-1/2} \in (\widehat{T^*}^{I^*})_{\Phi^*}$ has a canonical lift belonging to $\widehat{T^*}^{\Gamma^*}$. So applying the norm N, we obtain

$$(\delta_{B_{M^*}}^{-1/2})^f N(z_1^*) = ({}^{w^*}\delta_{B_{M^*}}^{-1/2})^f\, n^* N(z_2^*)\, n^{*-1}. \tag{2.3}$$

By construction $N(z_1^*) \in \mathfrak{T}^*$, and $n^* N(z_2^*) n^{*-1}$ belongs to the analogous open set in $Z({}^{w^*}\widehat{M^*})^{\Gamma^*}$. Recall that $\bar{\alpha}_i(\delta_{B_{M^*}}^{-1/2}) = q$ and $\bar{\beta}_j(\delta_{B_{M^*}}^{-1/2}) = 1$ for all i, j. Using this and Definition 2.3(a), $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\}$ is precisely the set of roots in $\Phi(\widehat{G^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ})$ whose value on the left hand side of (2.3) is q^f. Similarly, $\{w^*(\bar{\alpha}_1), \ldots, w^*(\bar{\alpha}_l)\}$ is the set of roots whose value on the right hand side of (2.3) is q^f. Therefore, these two sets coincide, in other words

$$\Delta(\widehat{M^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma^*,\circ}) = \Delta({}^{n^*}\widehat{M^*}^{\Gamma^*,\circ}, \widehat{T^*}^{\Gamma,\circ}).$$

Since n^* is fixed by Γ^*, it follows from this that $\Delta(\widehat{M^*}, \widehat{T^*}) = \Delta({}^{n^*}\widehat{M^*}, \widehat{T^*})$. Indeed, for any simple root $\alpha_i \in \Delta(\widehat{M^*}, \widehat{T^*})$, if ${}^{w^*}\alpha_i$ involves roots outside $\Delta(\widehat{M^*}, \widehat{T^*})$, then the Γ^*-average of ${}^{w^*}\alpha_i$ would involve roots outside $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\}$.

Therefore, we see that n^* normalizes $\widehat{M^*}$, in a positive-root preserving way, so that

$$\delta_{B_{M^*}}^{-1/2} = {}^{w^*}\delta_{B_{M^*}}^{-1/2}. \tag{2.4}$$

Further, by Remark 1.2, we see that the coset of $W(G^*, S^*)/W(M^*, S^*)$ corresponding to w^* is the image of Φ-fixed coset in $W(G, S)/W(M, S)$, that is, w^* comes from an element

$w \in W(G, A)$. Then (2.2) and (2.4) imply

$$z_1 = wz_2w^{-1}$$

as desired. ■

2.3 Proof of Lemma 8.2′

Proof. We consider the following commutative diagram

$$\begin{tikzcd} (Z(\widehat{M})^{I_F})_{\Phi_F} \arrow[r, "i"] \arrow[d] & (\widehat{T^*}^{I^*_F})_{\Phi^*_F} \arrow[d] \\ (Z(\widehat{M})^{I_F})_{\Phi_F}/W(G,A) \arrow[r, "\bar{i}"] & (\widehat{T^*}^{I^*_F})_{\Phi^*_F}/W(G^*,A^*). \end{tikzcd} \tag{2.5}$$

Here for brevity we write i for the map $\tilde{\psi}_0$ and $\bar{i}$ for $\bar{\psi}_0$. As remarked above in Section 1, the mistake in [4, Lemma 11.12.3] does not affect the subsequent result [4, Lemma 11.12.4], which shows that i induces the bottom arrow $\bar{i}$. Also, the valid part of the proof of [5, Lemma 8.2] shows that i is a closed immersion.

We prove that $\bar{i}$ is a finite morphism. On the level of $\mathbb{C}$-algebras, we have a commutative diagram

$$\begin{tikzcd} \mathbb{C}[(Z(\widehat{M})^{I_F})_{\Phi_F}] & \mathbb{C}[(\widehat{T^*}^{I^*_F})_{\Phi^*_F}] \arrow[l, two heads, "i^*"'] \\ \mathbb{C}[(Z(\widehat{M})^{I_F})_{\Phi_F}]^{W(G,A)} \arrow[u, hook] & \mathbb{C}[(\widehat{T^*}^{I^*_F})_{\Phi^*_F}]^{W(G^*,A^*)} \arrow[l, "\bar{i}^*"'] \arrow[u, hook] \end{tikzcd} \tag{2.6}$$

A diagram chase shows that the bottom arrow is finite-type and integral, hence finite.

Write $D = (Z(\widehat{M})^I)_\Phi$ and let D° be its neutral component. Then $X^*(D/D^\circ)$ is the torsion subgroup of $X^*(D)$. By [4, Lemma 11.3.4(b)], $W(G, A)$ acts trivially on $X^*(D/D^\circ)$ and hence it preserves the connected (=irreducible) components of $(Z(\widehat{M})^I)_\Phi$.

Let $\mathfrak{T}_0 := \hat{\psi}_0^{-1}\mathfrak{T}_*$, an open dense subset of D (cf. Def. 2.3). The image of $\mathfrak{T}_0$ in $D/W(G, A)$, call it $\bar{\mathfrak{T}}_0$, is open and meets every connected (=irreducible) component $C = zD^\circ/W(G, A)$. The irreducible components of $\mathrm{im}(\bar{i})$ are the sets $\bar{i}(C)$. By constructibility of the map $\bar{i}|_C$, there is an open dense subset $\mathfrak{T}_C \subset \bar{i}(C)$ contained in $\bar{i}(C \cap \bar{\mathfrak{T}}_0)$. Lemma 2.4 implies that $\bar{i}(C) \neq \bar{i}(C')$ if $C \neq C'$. Therefore, after possibly shrinking each $\mathfrak{T}_C$, we may assume $\mathfrak{T}_C \cap \bar{i}(C') = \emptyset$ whenever $C \neq C'$. Define $\mathfrak{T} := \cup_C \mathfrak{T}_C$ (a disjoint union); it is an open and dense subset of $\mathrm{im}(\bar{i})$. Also, the inverse image $\bar{i}^{-1}(\mathfrak{T})$ meets every irreducible component, hence is *dense* and open in D.

We may further shrink $\mathfrak{T}$ so that it is contained in the nonsingular locus of $\mathrm{im}(\bar{i})$, and the composition (2.5) is étale over $\mathfrak{T}$. Lemma 2.4 shows that $\bar{i}$ is injective on $\bar{i}^{-1}(\mathfrak{T})$. In fact $\bar{i} : \bar{i}^{-1}(\mathfrak{T}) \to \mathfrak{T}$ is a bijective finite étale morphism between non-singular finite-type $\mathbb{C}$-schemes. It is necessarily an isomorphism, and therefore $\bar{i}$ is birational on to its image. ■

Remark 2.5. The above proof shows that $\bar{i}$ induces a bijective map from the connected (=irreducible) components of D to the irreducible components of $\bar{i}(D)$. However, it seems quite possible that D could have more connected components than $\bar{i}(D)$. □

2.4 Example I: $G^* = \mathrm{GL}_n$

Our goal here is to prove the following.

Lemma 2.6. If $G = \mathrm{GL}_r(D)$ is an F-inner form of $G^* = \mathrm{GL}_n$, then the morphism $\bar{\psi}_0$ is a closed immersion. □

Proof. The argument is based on that of Jon Cohen [1, Theorem 5.3].

We have $n = rd$, where $[D : F] = d^2$. We identify $\widehat{G^*} = \mathrm{GL}_n(\mathbb{C})$. If M is a minimal F-Levi subgroup of G, we may identify $\widehat{M} = \widehat{M^*}$ with the "diagonal" Levi which is the product r copies of GL_d on the main diagonal. Let $\widehat{T^*}$ be the diagonal maximal torus in $\widehat{M^*}$, whose elements we can represent as tuples $(x_{10}, \dots, x_{1\,d-1}, \dots\dots, x_{r0}, \dots, x_{r\,d-1})$, where $x_{ij} \in \mathbb{C}^\times$ for $1 \leq i \leq r$ and $0 \leq j \leq d-1$. Similarly $Z(\widehat{M}) \cong \{(y_1, \dots, y_r) \,|\, y_i \in \mathbb{C}^\times\}$. The embedding $i : Z(\widehat{M}) \to \widehat{T^*}$ in these coordinates is

$$(y_1, \dots, y_r) \longmapsto (q^{\frac{d-1}{2}} y_1, \dots, q^{-\frac{d-1}{2}} y_1, \dots\dots, q^{\frac{d-1}{2}} y_r, \dots, q^{-\frac{d-1}{2}} y_r).$$

We may identify the map

$$\mathbb{C}[\widehat{T^*}] \xrightarrow{i^*} \mathbb{C}[Z(\widehat{M})]$$

with the map

$$\begin{aligned} \mathbb{C}[X_{10}^{\pm 1}, \dots, X_{r\,d-1}^{\pm 1}] &\to \mathbb{C}[Y_1^{\pm 1}, \dots, Y_r^{\pm 1}] \\ X_{ij} &\mapsto q^{\frac{d-1}{2} - j}\, Y_i. \end{aligned}$$

We can regard $W(G,A) = S_r$ as a subgroup of $W(G^*,A^*) = S_{rd}$ (i.e., as the subgroup of $r\times r$ "elementary matrices" whose "entries" are $d \times d$ identity or zero matrices). Therefore, on taking invariants we have the map

$$\begin{aligned} \mathbb{C}[X_{10}^{\pm 1},\dots,X_{r\,d-1}^{\pm 1}]^{S_{rd}} &\overset{\bar{i}^*}{\to} \mathbb{C}[Y_1^{\pm 1},\dots,Y_r^{\pm 1}]^{S_r} \\ p_k(X_{ij}) &\mapsto q^{\frac{d-1}{2}k}\cdot\frac{1-q^{-kd}}{1-q^{-k}}\cdot p_k(Y_i), \end{aligned} \tag{2.7}$$

where $p_k(X_{ij})$ and $p_k(Y_i)$ are the k-th power sum symmetric polynomials in the given variables. Clearly (2.7) is a surjective homomorphism, and so $\bar{i}$ is a closed immersion. ■

2.5 Example II: $G^* = \mathrm{Sp}(4n)$

Our goal here is to prove that $\bar{\psi}_0$ is *not injective* when $G^* = \mathrm{Sp}(4n)$ for $n \geq 1$ and $G \ncong G^*$. First we need to construct the unique non-quasisplit inner form G of G^*. Let $(D,-)$ be a quaternionic algebra over F endowed with an involution $d \mapsto \bar{d}$ of the first kind (fixing F). Define an involution $X \mapsto X^* := \bar{X}^t$ of the central simple F-algebra $M_{2n}(D)$. Now choose any $j \in D^\times$ with $\bar{j} = -j$, and define $J \in M_{2n}(D)$ by

$$J = \mathrm{antidiag}(j,j,\dots,j)$$

(the matrix with $2n$ copies of j on the antidiagonal, and zeroes elsewhere). Note that $J^* = -J$. Define the involution of $M_{2n}(D)$

$$\sigma\,:\,X \mapsto J^{-1}X^*J.$$

Finally define the group $G := \{X \mid \sigma(X)X = \mathrm{I}_{2n}\}$, where I_{2n} in the unit element of $M_{2n}(D)$. Since over a splitting field $X \mapsto X^*$ becomes isomorphic to the usual transpose operation and J becomes skew-symmetric, we see that G is an inner form of G^*.

A maximal F-split torus $A \subset G$ is of the form $A = \{\mathrm{diag}(a_1,\dots,a_n,a_n^{-1},\dots,a_1^{-1})\}$ where each a_i belongs to $F^\times$, the center of $D^\times$. Its centralizer M is a minimal F-Levi which geometrically is isomorphic to GL_2^n; it corresponds to the set of absolute simple roots

$$\Delta(M,T) = \{\alpha_1 = e_1 - e_2,\ \alpha_3 = e_3 - e_4,\ \dots,\ \alpha_{2n-1} = e_{2n-1} - e_{2n}\}$$

$$\Delta(G,T)\backslash\Delta(M,T) = \{\alpha_2 = e_2 - e_3,\ \dots,\ \alpha_{2n-2} = e_{2n-2} - e_{2n-1},\ \alpha_{2n} = 2e_{2n}\}.$$

On the dual side, we identify $\widehat{M} = \widehat{M^*}$ (the Galois action is trivial since $G^* = \mathrm{Sp}(4n)$ is split). The dual Levi $\widehat{M}$ corresponds to

$$\Delta(\widehat{M}, \widehat{T}) = \{\alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{2n-1}^\vee = e_{2n-1} - e_{2n}\}$$
$$\Delta(\widehat{G}, \widehat{T}) \backslash \Delta(\widehat{M}, \widehat{T}) = \{\alpha_2^\vee = e_2 - e_3, \ldots, \alpha_{2n-2}^\vee = e_{2n-2} - e_{2n-1}, \alpha_{2n}^\vee = e_{2n}\}.$$

We will construct $w \in W(G^*, A^*)$ and $z \in Z(\widehat{M})$ with $z \neq 1$ and with

$$w(\delta_{B_{M^*}}^{-1/2} z) w^{-1} = \delta_{B_{M^*}}^{-1/2}. \tag{2.8}$$

Clearly such a relation proves that $\bar{\psi}_0$ is not injective. We take $w = \epsilon_{2n-1} s_{e_{2n-1} - e_{2n}}$ (so e.g., $w(e_{2n-1} - e_{2n}) = e_{2n-1} + e_{2n}$). Now assume $n > 1$. Then

$$w(\alpha_i^\vee) = \alpha_i^\vee, \ 1 \leq i \leq 2n-3$$
$$w(\alpha_{2n-2}^\vee) = \alpha_{2n-2}^\vee + \alpha_{2n-1}^\vee$$
$$w(\alpha_{2n-1}^\vee) = \alpha_{2n-1}^\vee + 2\alpha_{2n}^\vee$$
$$w(\alpha_{2n}^\vee) = -(\alpha_{2n-1}^\vee + \alpha_{2n}^\vee).$$

We can make (2.8) hold by choosing $z \in Z(\widehat{M})$ with $\alpha_2^\vee(z) = \alpha_4^\vee(z) = \cdots = \alpha_{2n-4}^\vee(z) = 1$, $\alpha_{2n-2}^\vee(z) = q$, and $\alpha_{2n}^\vee(z) = q^{-1}$.

If $n = 1$, a similar argument shows we can make (2.8) hold if we require that z satisfies $\alpha_2^\vee(z) = q^{-1}$. ■

Remark 2.7. Any finite surjective birational morphism of complex varieties $f : X \to Y$ with X irreducible and Y normal, must be an isomorphism. It follows that $\mathrm{im}(\bar{\psi}_0)$ is not normal in the example just considered. □

2.6 Impact on other results of [5]

We use the notation of [5]. Here is a list of the necessary changes:

- When G is general non-quasisplit, and $K \subset G(F)$ is a special maximal parahoric subgroup, all references to a *parametrization* $\Pi(G, K) \xrightarrow{\sim} S(G)$ should be replaced with a *finite-to-one map* $\Pi(G, K) \to S(G)$. This appears in Theorem. 1.1(A) and Theorem. 10.1. (If $G^* = \mathrm{GL}_n$, then no changes are necessary thanks to Section 2.4.)
- One should delete Section 12 on the map $\Pi(G, K) \to \Pi(G^*, K^*)$.

- Lemma 8.2 should be replaced with Lemma 8.2′ above, and Corollary 8.4 should be deleted. The reason the proof of Lemma 8.2 fails is that the map (8.5) need not be surjective (cf. Lemma 11.12.3′).
- The maps (1.3) and (9.1) are not always injective; in Definition 9.1, the map $\tilde{t}_{A^*A}$ is not always injective.

Funding

This work was partially supported by the National Science Foundation [DMS-1406787 to T.J.H.] and by a 2017 Simons Fellowship in Mathematics.

Acknowledgments

I thank Jon Cohen for a useful conversation around the subject of this note. I am very grateful to Timo Richarz for his helpful comments and questions on the manuscript.

References

[1] Cohen, J. "Transfer of representations and orbital integrals for inner forms of GL_n." arXiv:1606.02141.

[2] Digne, F. and J. Michel. "Representations of Finite Groups of Lie Type." London Math. Soc. Student Texts, vol. 21, 161 pp. Cambridge: Cambridge Univ. Press. 1991.

[3] Haines, T. "Dualities for root systems with automorphisms and applications to non-split groups." To appear in *Representation Theory*, arXiv:1604.01468.

[4] Haines, T. "The Stable Bernstein Center and Test Functions for Shimura Varieties." In *Automorphic Forms and Galois Representations,* edited by F. Diamond, P. Kassaei and M. Kim, vol. 2, 118–86. London Math. Soc. Lecture Notes, Series 415. Cambridge: Cambridge University Press, 2014.

[5] Haines, T. "On Satake parameters for representations with parahoric fixed vectors." *International Mathematical Research Notices* no. 20 (2015): 10367–98.

[6] Haines, T. and S. Rostami. "The Satake isomorphism for special maximal parahoric Hecke algebras." *Representation Theory* 14 (2010): 264–84.

[7] Kottwitz, R. "Stable trace formula: cuspidal tempered terms." *Duke Mathematical Journal* 55, no. 3 (1984): 611–50.