ON SATAKE PARAMETERS FOR REPRESENTATIONS WITH PARAHORIC FIXED VECTORS

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Abstract. This article, a continuation of [HRo], constructs the Satake parameter for any irreducible smooth J-spherical representation of a p-adic group, where J is any parahoric subgroup. This parametrizes such representations when J is a special maximal parahoric subgroup. The main novelty is for groups which are not quasi-split, and the construction should play a role in formulating a geometric Satake isomorphism for such groups over local function fields.

1. Introduction

Let F be a nonarchimedean local field and let W_F denote its Weil group, with I_F its inertia subgroup and $\Phi \in W_F$ a choice of a geometric Frobenius element. Let G be a connected reductive group over F, with complex dual group $\widehat{G}$. Let $J \subset G(F)$ be a parahoric subgroup, and let $\Pi(G/F, J)$ denote the set of isomorphism classes of smooth irreducible representations π of $G(F)$ such that $\pi^J \neq 0$. To $\pi \in \Pi(G/F, J)$ we will associate a *Satake parameter* $s(\pi)$ belonging to the variety $[\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$, where the quotient is formed using the conjugation action of $\widehat{G}^{I_F}$ on the set of semisimple elements in the coset $\widehat{G}^{I_F} \rtimes \Phi$. More precisely, we will prove the following theorem.

Theorem 1.1. *There is an explicit closed subvariety $S(G) \subseteq [\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$ and a canonical map $s : \Pi(G/F, J) \to S(G)$ with the following properties:*

(A) *If $J = K$ is a special maximal parahoric subgroup, the map $\pi \mapsto s(\pi)$ gives a* parametrization
$$\Pi(G/F, K) \xrightarrow{\sim} S(G).$$
(B) *$S(G) = [\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$ if and only if G/F is quasi-split.*

(C) *The parameter $s(\pi)$ predicts part of the local Langlands parameter φ_π that is conjecturally attached to π: $\varphi_\pi(\Phi) = s(\pi)$ in $[\widehat{G} \rtimes \Phi]_{\rm ss}/\widehat{G}$ (Conjecture 13.1).*

The evidence for (C) is contained in the following result, which we prove in §13, under the assumption that inner forms of GL_n satisfy the enhancement LLC+ of the local Langlands correspondence (see [H13, §5.2]).

Theorem 1.2. *Conjecture 13.1 holds if G is any inner form of GL_n.*

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The map $\pi \mapsto s(\pi)$ is constructed as follows: to π we associate its supercuspidal support, which by [H13, §11.5] is a cuspidal pair $(M, \chi)_G$ with $M = \mathrm{Cent}_G(A)$ a minimal F-Levi subgroup of G and $\chi \in X^{\mathrm{w}}(M) = \mathrm{Hom}_{\mathrm{grp}}(M(F)/M(F)_1, \mathbb{C}^\times)$ a weakly unramified character on $M(F)$ (in the terminology of [H13, 3.3.1]). Here $M(F)_1$ is the kernel of the Kottwitz homomorphism [Ko97, §7], the theory of which gives an isomorphism

$$\kappa_M : M(F)/M(F)_1 \xrightarrow{\sim} X^*(Z(\widehat{M})^{I_F}_{\Phi}). \tag{1.1}$$

Recalling that χ is determined by π up to conjugation by the relative Weyl group $W(G, A)$, we can view the supercuspidal support of π as an element in the complex affine variety $(Z(\widehat{M})^{I_F})_\Phi/W(G, A)$. Thus $\pi \mapsto \chi$ gives a map

$$\Pi(G/F, J) \to (Z(\widehat{M})^{I_F})_\Phi/W(G, A). \tag{1.2}$$

On the other hand, if (G, Ψ) is an F-inner form of a quasi-split group G^*, and if $A^* \subset T^* \subset G^*$ are data parallel to $A \subset M \subset G$, then the theory of the normalized transfer homomorphisms $\tilde{t}_{A^*,A}$ from §8 together with the material in §5, 6 gives rise to a canonical closed immersion

$$(Z(\widehat{M})^{I_F})_\Phi/W(G, A) \xhookrightarrow{(9.1)} [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}. \tag{1.3}$$

As explained in §9, the composition s of (1.2) with (1.3) is completely canonical (independent of the choice of A), and $S(G)$ is defined to be its image.

The following result gives two more conceptual descriptions of $S(G)$. Let $\mathcal{N}(\widehat{G}^{I_F})$ denote the set of nilpotent elements in $\mathrm{Lie}(\widehat{G}^{I_F})$. We call $(\hat{g} \rtimes \Phi, x) \in [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}} \times \mathcal{N}(\widehat{G}^{I_F})$ a *regular* Φ-*admissible pair* if

- $\mathrm{Ad}(\hat{g} \rtimes \Phi)(x) = q_F\, x$, where q_F is the cardinality of the residue field of F;
- $\Phi(x) = x$;
- x is a *principal* nilpotent element in $\mathrm{Lie}(\widehat{G}^{I_F})$.

Denote the set of such pairs by $\mathcal{P}^{\Phi}_{\mathrm{reg}}(\widehat{G}^{I_F})$.

Theorem 1.3. *Let $M = \mathrm{Cent}_G(A)$ be any minimal F-Levi subgroup of G, and let $\widehat{M} \subset \widehat{G}$ be a corresponding W_F-stable Levi subgroup of the complex dual group $\widehat{G}$. The following are equivalent for an element $\hat{g} \rtimes \Phi \in [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$:*

(i) *$\hat{g} \rtimes \Phi \in S(G)$;*
(ii) *$\hat{g} \rtimes \Phi$ is $\widehat{G}^{I_F}$-conjugate to the first coordinate of a pair in $\mathcal{P}^{\Phi}_{\mathrm{reg}}(\widehat{M}^{I_F})$;*
(iii) *$\hat{g} \rtimes \Phi$ is $\widehat{G}^{I_F}$-conjugate to the image under the natural map*

$$[\widehat{M^*}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{M^*}^{I_F} = [\widehat{M}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{M}^{I_F} \to [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$$

of the Satake parameter $s(\sigma^*)$ of some weakly unramified twist σ^* of the Steinberg representation for M^*. Here M^* is a quasi-split F-inner form of M (see Remark 5.3).

In formulating (ii) we used implicitly that $\widehat{G}^{I_F}$ and $\widehat{M}^{I_F}$ are reductive groups and that $\widehat{G}^{I_F} = Z(\widehat{G})^{I_F}\, \widehat{G}^{I_F,\circ}$; see §§4, 5.

Let us consider some special cases and history. The most important case is where $J = K$ is a special maximal parahoric subgroup. If G/F is unramified, then such a K is automatically a special *maximal compact* subgroup (cf. [HRo]), and $S(G) = [\widehat{G} \rtimes \Phi]_{\rm ss}/\widehat{G}$, and the parametrization in (A) is classical (cf. [Bor]). If G/F is only quasi-split and tamely ramified (i.e. split over a tamely ramified extension of F), then the parametrization in (A) was proved by M. Mishra [Mis] and some similar results were also obtained by X. Zhu [Zhu].

The same ideas show how to construct the s-parameter in the hypothetical Deligne-Langlands triple (s, u, ρ) one could hope to associate to parahoric-spherical representations of general connected reductive groups; see §13, where item (C) is also explained. It should be stressed that throughout this article, "parahoric" should be understood in the sense of Bruhat-Tits [BT2], as the $\mathcal{O}_F$-points of a *connected* group scheme over $\mathcal{O}_F$. So for example an Iwahori subgroup here is somewhat smaller than the "naive" notion that sometimes appears in the literature under the same name, and therefore the Iwahori-Hecke algebra and its center are slightly larger (cf. [H09c] and [H13, Appendix]).

It was clear that some kind of parametrization like that in (A) should hold, after the author and S. Rostami proved in [HRo] the general form of the Satake isomorphism

$$\mathcal{H}(G(F), K) \cong \mathbb{C}[(Z(\widehat{M})^{I_F})_\Phi / W(G, A)]. \tag{1.4}$$

It was also clear at that time that this isomorphism is the right one to "categorify", in other words it should be the function-theoretic shadow of a geometric Satake isomorphism à la [MV] for G/F and K, once such an isomorphism is properly formulated (of course here we assume $F = \mathbb{F}_q(\!(t)\!)$). In the meantime, progress in exactly this direction has been made: X. Zhu [Zhu] proved a geometric Satake isomorphism extending (1.4) for quasi-split and tamely ramified G (and *very special* K, in Zhu's terminology). This was recently generalized by T. Richarz [Ri], who effectively removed the "tamely ramified" hypothesis from Zhu's result, while still assuming G is quasi-split and K is very special.

One obstacle to formulating a geometric Satake isomorphism when G/F is not quasi-split is the lack of a suitable link between the right hand side of (1.4) and the L-group ${}^LG := \widehat{G} \rtimes W_F$. We are proposing that (1.3) provides the sought-after link. In order to fully justify this idea, it would be important to establish a suitable "categorification" of the normalized transfer homomorphisms, of the subvariety $S(G)$, and of the closed immersion (1.3). The author hopes to return to these matters in future work.

Here is an outline of the contents of this article. In §2 we recall some notation that is used throughout the paper. In §3 we recall the parametrization of $\Pi(G/F, K)$ that is a consequence of (1.4) and other results from [HRo]. The purpose of §4 is to lay some groundwork needed in order to prove properties of $\widehat{G}^{I_F}$ (e.g. it is reductive; analysis of its group of connected components) which are needed in §§5, 6, 7 on the parameter space

$[\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$. Those sections handle the construction of $\pi \mapsto s(\pi)$ when G/F is quasi-split. Section §8 provides the key ingredients (transfer homomorphisms, etc.) needed to extend the construction to the general case, which is done in §§9, 10. Theorem 1.1 parts (A) and (B) are proved in §10. We prove Theorem 1.3 in §11, relying on the key Lemma 4.9 proved at the end of §4. Finally, in §§12, 13 we explain the connection of the Satake parameters to the (conjectural) local Langlands and Jacquet-Langlands correspondences, and also justify (C) by proving Theorem 1.2.

2. Notation and conventions

We denote the absolute Galois group of F by $\Gamma := \mathrm{Gal}(F^s/F)$, where F^s is some separable closure of F, fixed once and for all.

If G is any connected reductive group over a nonarchimedean field F, and if $J \subset G(F)$ is any compact open subgroup, then $\mathcal{H}(G(F), J) := C_c(J\backslash G(F)/J)$, a $\mathbb{C}$-algebra when endowed with the convolution $*$ defined by using the Haar measure on $G(F)$ which gives J volume 1. We write $\mathcal{Z}(G(F), J)$ for the center of $\mathcal{H}(G(F), J)$.

For any F-Levi subgroup M and F-parabolic subgroup P with unipotent radical N and Levi decomposition $P = MN$, we define for $m \in M(F)$ the usual modulus function

$$\delta_P(m) := |\det(\mathrm{Ad}(m); \mathrm{Lie}\, N(F))|_F,$$

where $|\cdot|_F$ is the normalized absolute value on F. Then for any admissible representation σ of $M(F)$, we set $i_P^G(\sigma) := \mathrm{Ind}_{P(F)}^{G(F)}(\sigma \otimes \delta_P^{1/2})$ where $\mathrm{Ind}_?^?(?)$ denotes usual (unnormalized) induction.

We use ${}^xY$ to denote xYx^{-1} for x an element and Y a subset of some group. If f is a function on that group, ${}^xf$ will be the function $y \mapsto f(x^{-1}yx)$.

We will use Kottwitz' conventions on dual groups $\widehat{G}$ and their Γ-actions, see [Ko84, §1].

3. First parametrization of K-spherical representations

Fix a special maximal parahoric subgroup $K \subset G(F)$. In G, choose any maximal F-split torus A whose associated apartment in the Bruhat-Tits building $\mathcal{B}(G_{\rm ad}, F)$ contains the special vertex associated to K. Let $M := \mathrm{Cent}_G(A)$ be the centralizer of A, a minimal F-Levi subgroup. Following [H13], we call the group of homomorphisms $M(F)/M(F)_1 \to \mathbb{C}^\times$ the group $X^{\rm w}(M)$ of *weakly unramified* characters on $M(F)$. The Kottwitz homomorphism [Ko97, §7] induces an isomorphism $M(F)/M(F)_1 \cong X^*(Z(\widehat{M})^{I_F}_\Phi)$, so that $X^{\rm w}(M) \cong (Z(\widehat{M})^{I_F})_\Phi$, a diagonalizable group over $\mathbb{C}$.

Given $\chi \in X^{\rm w}(M)$, the Iwasawa decomposition of [HRo, Cor. 9.1.2] allows us to define an element $\Phi_{K,\chi} \in i_P^G(\chi)^K$ by

$$\Phi_{K,\chi}(mnk) = \delta_P^{1/2}(m)\, \chi(m)$$

for $m \in M(F)$, $n \in N(F)$, and $k \in K$ (here N is the unipotent radical of an F-parabolic subgroup P having M as Levi factor). Then define the spherical function

$$\Gamma_\chi(g) = \int_K \Phi_{K,\chi}(kg)\, dk$$

where $\mathrm{vol}_{dk}(K) = 1$. Let π_χ denote the smallest G-stable subspace of the right regular representation of $G(F)$ on $C^\infty(G(F))$ containing Γ_χ. Then, as in [Car, §4.4], we see that π_χ is irreducible, that $\pi_\chi \cong \pi_{\chi'}$ iff $\chi = {}^w\chi'$ for some $w \in W(G, A)$, and that every element of $\Pi(G/F, K)$ is isomorphic to some π_χ. Thus we have the following first parametrization of $\Pi(G/F, K)$.

Proposition 3.1. *The map $\chi \mapsto \pi_\chi$ sets up a 1-1 correspondence*

$$(Z(\widehat{M})^{I_F})_\Phi / W(G, A) \xrightarrow{\sim} \Pi(G/F, K). \tag{3.1}$$

Moreover, if $f \in \mathcal{H}(G(F), K)$, then $\pi_\chi(f)$ acts on π_χ^K by the scalar $S(f)(\chi)$, where S is the Satake isomorphism

$$S : \mathcal{H}(G(F), K) \xrightarrow{\sim} \mathbb{C}[(Z(\widehat{M})^{I_F})_\Phi / W(G, A)] \tag{3.2}$$

of [HRo, Thm. 1.0.1]. *Here the right hand side denotes the ring of regular functions on the affine variety $(Z(\widehat{M})^{I_F})_\Phi / W(G, A)$.*

4. Fixed-point subgroups under finite groups of automorphisms

Steinberg [St] proved fundamental results on cyclic groups of automorphisms of a simply connected semisimple algebraic group. In the same context, when the generator of the cyclic group comes from a diagram automorphism, Springer [Sp2] supplemented Steinberg's results by, among other things, giving information about the root data of the fixed-point group. The aim here is to extend some of the results of Steinberg and Springer to finite groups of automorphisms of a reductive group. The following might be known, but we include a complete proof here due to the lack of a suitable reference.

Notation: If a group J acts by automorphisms on an algebraic group P, we write P° for the neutral component of P and often write $P^{J,\circ}$ instead of $(P^J)^\circ$.

Proposition 4.1. *Let H be a possibly disconnected reductive group over an algebraically closed field k. Assume that a finite group I acts by automorphisms on H and preserves a splitting (T, B, X), consisting of a Borel subgroup B, a maximal torus T in B, and a principal nilpotent element $X = \sum_{\alpha \in \Delta(T,B)} X_\alpha$ for some non-zero elements $X_\alpha \in (\mathrm{Lie}\, H^\circ)_\alpha$ indexed by the B-positive simple roots $\Delta(T, B)$ in $X^*(T)$. Let U be the unipotent radical of B, and let $N = N(H^\circ, T)$ be the normalizer of T in H°. Then:*

- (a) *The algebraic group H^I is reductive with identity component $(H^I)^\circ = [(H^\circ)^I]^\circ$ and with splitting $(T^{I,\circ}, B^{I,\circ}, X^I)$, where $B^{I,\circ} = T^{I,\circ}U^I$ and where X^I is a principal nilpotent in $\mathrm{Lie}(H^{I,\circ})$ constructed from X. (If $\mathrm{char}(k) \neq 2$, then $X^I = X$.)*
- (b) *We have $T^I \cap H^{I,\circ} = T^{I,\circ}$ and $N^I \cap H^{I,\circ} = N(H^{I,\circ}, T^{I,\circ})$.*

(c) *If* $W := W(H^\circ, T) := N/T$, *then every element of* W^I *has a representative in* $N^I \cap H^{I,\circ}$, *and thus* $W(H^{I,\circ}, T^{I,\circ}) = W^I$.

(d) *The inclusion* $T \hookrightarrow H^\circ$ *induces a bijection* $\pi_0\, T^I \,\tilde{\to}\, \pi_0\,(H^\circ)^I$.

Before beginning the proof, note that giving the data of X is equivalent to giving the data $\{x_\alpha\}_{\alpha \in \Delta(T,B)}$ of root group homomorphisms $x_\alpha : \mathbb{G}_a \,\tilde{\to}\, U_\alpha$, where U_α is the maximal connected unipotent subgroup of H° normalized by T and with $\mathrm{Lie}(U_\alpha) = (\mathrm{Lie}\, H^\circ)_\alpha$. This is because the Lie functor gives an isomorphism $\mathrm{Isom}_{k-\mathrm{Grp}}(\mathbb{G}_a, U_\alpha) = \mathrm{Isom}_k(k, \mathrm{Lie}(U_\alpha))$.

Proof. Quite generally $(H^I)^\circ$ contains $[(H^\circ)^I]^\circ$ with finite index, and as both are connected algebraic groups, they coincide.

Lemma 4.2. *Let* Ψ *be a reduced root system in a real vector space* V, *with set of simple roots* Δ. *Suppose* I *is a finite group of automorphisms of* V *which preserves* Ψ *and* Δ. *Let* $\bar{\alpha} \in V$ *denote the average of the* I*-orbit of* $\alpha \in \Psi$, *and let* $\Psi^I = \{\bar{\alpha} \mid \alpha \in \Psi\}$ *and* $\Delta^I = \{\bar{\alpha} \mid \alpha \in \Delta\}$. *Then*

(1) Ψ^I *is a possibly non-reduced root system in* V^I *with set of simple roots* Δ^I;

(2) $W(\Psi^I) = W(\Psi)^I$, *where* $W(\Sigma)$ *denotes the Weyl group of a root system* Σ.

Proof. First assume $I = \langle \tau \rangle$. Let $\Psi_\tau \subseteq \Psi^\tau$ be defined by discarding those elements of Ψ^τ which are smaller multiples of others. Then [St, 1.32, 1.33] shows that Ψ_τ is a root system with Weyl group $W(\Psi)^\tau$. The only difference between Ψ_τ and Ψ^τ is that the latter could contain $\frac{1}{2}\alpha'$ for $\alpha' \in \Psi_\tau$, and then only for a component of Ψ of type A_{2n}. Consideration of the root system for a quasi-split unitary group in $2n+1$ variables attached to a separable quadratic extension of a p-adic field (cf. [Tits, §1.15]) shows that adding such half-roots to a root system of form Ψ_τ, still gives a root system; so Ψ^τ is indeed a root system, with simple roots Δ^τ and with the same Weyl group $W(\Psi)^\tau$. Note that if $|\tau|$ is odd, then Ψ^τ is again reduced (comp. [HN, Lemma 9.2]).

Now decompose (V, Ψ) into a sum of simple systems (V_j, Ψ_j). The action of I permutes these simple systems while the stabilizer of each component continues to act through its automorphism group. Therefore we may assume (V, Ψ) is simple. Using the classification, we may assume I acts through a faithful action of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, or S_3 on Δ. In the last case, $\Psi = D_4$ and the I-orbits on Ψ^+ and on Δ coincide with those of the subgroup $\mathbb{Z}/3\mathbb{Z} \subset S_3$. Thus we may assume I is cyclic, and we may apply the preceding paragraph. □

We will apply Lemma 4.2 when $\Psi = \Psi(H^\circ, T)$ (resp. $\Delta = \Delta(T, B)$), the set of roots (resp. B-simple roots) for T in $\mathrm{Lie}(H^\circ)$ in the vector space $V = X^*(T) \otimes \mathbb{R}$. The set of positive roots Ψ^+ is a disjoint union of subsets S_α, where S_α consists of all those positive roots whose projection to V^I is proportional to that of α (comp. [St, Thm. 8.2(2')]). We will always index S_α by a *minimal* element α in this set (use the usual partial order on positive roots). For example (cf. [St, Thm. 8.2(2')]), if $I = \langle \tau \rangle$, then S_α comes in two types:

(Type 1) S_α is a τ-orbit $\{\alpha, \tau\alpha, \dots\}$, no two of which add up to a root;

(Type 2) $S_\alpha = \{\alpha, \tau\alpha, \beta\}$, where $\beta := \alpha + \tau\alpha$ is a root; this occurs only in type A_{2n}.

For a root system of the form Ψ^τ, we write $(\Psi^\tau)^{\rm red}$ (resp. $(\Psi^\tau)_{\rm red}$)) for the root system we get by discarding vectors from Ψ^τ which are shorter (resp. longer) multiples of others. For example, $\Psi_\tau = (\Psi^\tau)^{\rm red}$.

We now make the following temporary assumptions:

(i) $H^{I,\circ}$ is reductive with splitting $(T^{I,\circ}, B^{I,\circ}, X^I)$, where X^I denotes a principal nilpotent element of ${\rm Lie}(H^{I,\circ})$ constructed from X. (If ${\rm char}(k) \neq 2$, then $X^I = X$.)

(ii) $\Psi(H^{I,\circ}, T^{I,\circ})$ can be identified with $(\Psi^I)_{\rm red}$ if ${\rm char}(k) \neq 2$ and with $(\Psi^I)^{\rm red}$ if ${\rm char}(k) = 2$.

(iii) If ${\rm char}(k) \neq 2$, the group U^I is the product of certain subgroups $U_{\bar\alpha}$ indexed by the various subsets S_α, where ${\rm Ad}(T^{I,\circ})$ acts via $\bar\alpha$ on ${\rm Lie}(U_{\bar\alpha}) \subset {\rm Lie}(H^{I,\circ})$. If $\alpha \in \Delta(T,B)$, then $U_{\bar\alpha} \cong \mathbb{G}_a$, and in general $U_{\bar\alpha}$ is either trivial or is isomorphic to $\mathbb{G}_a$. In particular U^I is connected. Furthermore, $U_{\bar\alpha}$ is contained in the product of the groups $U_{\alpha'}$ for $\alpha' \in S_\alpha$. If ${\rm char}(k) = 2$, the same statements hold with $U_{\bar\alpha}$ replaced by $U_{2\bar\alpha}$ when S_α is of Type 2.

Remark 4.3. Property (iii) automatically implies another property:

(iv) The map $N^I \to W(\Psi)^I$, $n \mapsto w_n$, has the property that for every subset S_α, we have $nU_{i\bar\alpha}n^{-1} = U_{w_n(i\bar\alpha)}$, for $i \in \{1,2\}$.

Remark 4.4. Later we will see that $U_{\bar\alpha}$ (resp. $U_{2\bar\alpha}$) is isomorphic to $\mathbb{G}_a$ for *all* $\alpha \in \Psi^+$ representing a set S_α, not just for $\alpha \in \Delta(T,B)$.

Lifting step. Let $s_{\bar\alpha} \in W(\Psi)^I$ be the reflection corresponding to a *simple* root $\bar\alpha$ for some $\alpha \in \Delta(T,B)$ (cf. Lemma 4.2). We wish to show it can be lifted to an element in $N^I \cap H^{I,\circ}$. Using (iii), we may choose $u \in U_{\bar\alpha}\backslash\{1\}$ if ${\rm char}(k) \neq 2$ or S_α is Type 1 (resp. $u \in U_{2\bar\alpha}\backslash\{1\}$ if ${\rm char}(k) = 2$ and S_α is Type 2). Using the Bruhat decomposition for $\overline{U}$ in place of U, we can write uniquely $u = u_1 n u_2$, where $u_2 \in \overline{U}$, $n \in N$, and $u_1 \in \overline{U} \cap nUn^{-1}$; since u is I-fixed, u_1, n, u_2 are too. The element n belongs to $N^I \cap \overline{U}^I U_{i\bar\alpha} \overline{U}^I$ $(i \in \{1,2\})$ and thus to $N^I \cap H^{I,\circ}$ by (iii). The element $w_n \in W \cong W(\Psi)$ to which n projects is different from 1, is fixed by I, and is in the group generated by the reflections $s_{\tau(\alpha)}$, $\tau \in I$. For the last statement, use (iii) and [Sp1, 9.2.1] to show that $n \in \langle U_{\pm\alpha'}\rangle_{\alpha'} \cdot \overline{U}$ where α' ranges over elements in S_α (a Levi subset of Ψ^+ when $\alpha \in \Delta$), and then use the Bruhat decomposition again. Let $\Psi^{I,+}$ denote the positive roots of Ψ^I. If $w_n \neq s_{\bar\alpha}$ then w_n sends some root in $\Psi^{I,+}\backslash\{\bar\alpha, 2\bar\alpha\}$ to $-\Psi^{I,+}$. Then as an element of $W(\Psi)$, w_n makes negative some positive root outside S_α, in violation of $w_n \in \langle s_{\tau(\alpha)}, \tau \in I\rangle$. Thus $w_n = s_{\bar\alpha}$, and n is the desired lift of $s_{\bar\alpha}$.

By Lemma 4.2, $W(\Psi)^I = W(\Psi^I)$ and so any element $w \in W(\Psi)^I$ is a product of elements $s_{\bar\alpha}$ as above; hence w can be lifted to $N^I \cap H^{I,\circ}$. This proves part of (c). Since N^I clearly maps to $W^I \cong W(\Psi)^I$, the proof also shows that $N^I \subset \langle T^I, U^I, \overline{U}^I\rangle$. As $(H^\circ)^I = \langle N^I, U^I\rangle$ by the Bruhat decomposition of H°, we obtain

$$(H^\circ)^I = \langle T^I, U^I, \overline{U}^I\rangle. \tag{4.1}$$

From this we see $[(H^\circ)^I]^\circ$ contains the connected subgroup $\langle (T^I)^\circ, U^I, \overline{U}^I\rangle$ with finite index, and so

$$[(H^\circ)^I]^\circ = \langle (T^I)^\circ, U^I, \overline{U}^I\rangle. \tag{4.2}$$

Hence

$$(H^\circ)^I = T^I \cdot [(H^\circ)^I]^\circ. \tag{4.3}$$

We claim that $(T^I)^\circ = T^I \cap [(H^\circ)^I]^\circ$. The inclusion "$\subseteq$" is clear. As for the other, using (4.2) it is enough to show that $T^I \cap \langle U^I, \overline{U}^I\rangle \subset (T^I)^\circ$. But $\langle U^I, \overline{U}^I\rangle$ lies in the image of $[(H^\circ)_{\rm sc}]^I \rightarrow [(H^\circ)_{\rm der}]^I$, and so we are reduced to the case where H° is simply connected.[1] In that case $X^*(T)$ has a $\mathbb{Z}$-basis permuted by I, and so T^I is already connected, and the result is obvious.

It is clear that $N^I \cap H^{I,\circ} \subseteq N(H^{I,\circ}, T^{I,\circ})$. We claim that equality holds. The Bruhat decomposition for the reductive group H° implies that an element of $[(H^\circ)^I]^\circ$ decomposes uniquely in the form unv where $n \in N^I \cap [(H^\circ)^I]^\circ$, $v \in U^I$ and $u \in U^I \cap {}^n\overline{U}^I$. (Note n automatically belongs to $[(H^\circ)^I]^\circ$ since U^I and $U^I \cap {}^n\overline{U}^I$ are connected, as follows from (iii, iv).) Since by (i,iii) $[(H^\circ)^I]^\circ$ is reductive with Borel subgroup $B^{I,\circ} = T^{I,\circ}U^I$, the element also decomposes as $u_1n_1v_1$ where $n_1 \in N(H^{I,\circ}, T^{I,\circ})$, $v_1 \in U^I$, and $u_1 \in U^I \cap {}^{n_1}\overline{U}^I$. Comparing these decompositions, we see $n = n_1$, i.e., $N(H^{I,\circ}, T^{I,\circ}) = N^I \cap H^{I,\circ}$.

As $N^I \cap H^{I,\circ}$ surjects onto W^I, we deduce $N^I \cap H^{I,\circ}/T^I \cap H^{I,\circ} \xrightarrow{\sim} W^I$. The above paragraphs show the left hand side is $W(H^{I,\circ}, T^{I,\circ})$.

At this point we have proved (a-d) assuming (i-iii).[2] Now we need to prove (i-iii). We first consider the case where I is generated by a single element τ. We use results of Steinberg [St], especially 8.2, 8.3. (Much of what we need also appears in [KS, §1.1].) We will adapt the proof of [St, Theorem 8.2]: it assumes H° is semisimple and simply connected and only assumes τ fixes T and B, but the argument carries over when τ fixes a splitting because in [St, Theorem 8.2] step (5) we may take $t = 1$. Indeed, for each B-positive root $\alpha \in X^*(T)$ let $x_\alpha : \mathbb{G}_a \rightarrow H^\circ$ be the corresponding root homomorphism, and write

$$\tau x_{\tau\alpha}(y) = x_\alpha(c_{\tau\alpha}y) \tag{4.4}$$

for all $y \in k$ and some constants $c_\alpha \in k^\times$. (Here $\tau\alpha$ is defined to be $\alpha \circ \tau : T \rightarrow \mathbb{G}_m$.) Then the hypothesis that τ fixes X is equivalent to $c_\alpha = 1$, $\forall \alpha \in \Delta(T, B)$. Since $\alpha(t) = c_{\tau\alpha}$ for $\alpha \in \Delta(T, B)$ by definition of t, we may choose $t = 1$.

Our first task is to prove (iii). Recall that [St] shows that U^τ is connected by analyzing the conditions under which an element of the form $x_\alpha(y_\alpha)x_{\tau\alpha}(y_{\tau\alpha})\cdots$ (indices ranging over S_α) belongs to U^τ. To ease notation, write H_1 (resp. T_1, B_1) for $H^{\tau,\circ}$ (resp. $T^{\tau,\circ}$, $B^{\tau,\circ}$). First suppose $\alpha \in \Psi^+$ represents a set S_α of Type 1. Consider the average $\bar{\alpha}$ of the orbit $\{\alpha, \tau\alpha, \dots\}$. By [St, Thm. 8.2 (2)], there are nontrivial elements in U^τ of the form

[1] Since it fixes a splitting, the action of I on $(H^\circ)_{\rm der}$ can be lifted to give a compatible action on its simply-connected cover $(H^\circ)_{\rm sc}$, for example by using the Isomorphism Theorem [Sp1, 9.6.2].

[2] In fact only (i,iii) are needed in the argument.

$x_\alpha(y_\alpha)x_{\tau\alpha}(y_{\tau\alpha})\cdots$ only if $c_\alpha c_{\tau\alpha}\cdots = 1$, in which case they form a 1-parameter subgroup $U_{\bar\alpha}$ consisting of elements of the form

$$x^{H_1}_{\bar\alpha}(y) = x_\alpha(y)x_{\tau\alpha}(c^{-1}_{\tau\alpha}y)\cdots \tag{4.5}$$

for $y \in k$. If $\alpha \in \Delta(T,B)$, then $c_\alpha = c_{\tau\alpha} = \cdots = 1$, so we have $U_{\bar\alpha} \cong \mathbb{G}_a$ and $x^{H_1}_{\bar\alpha}$ is given by the formula

$$x^{H_1}_{\bar\alpha}(y) = x_\alpha(y)\, x_{\tau\alpha}(y)\cdots . \tag{4.6}$$

Next suppose $\alpha \in \Psi^+$ represents a set S_α of Type 2: $S_\alpha = \{\alpha, \tau(\alpha), \beta\}$, where $\beta := \alpha + \tau\alpha$ is a root of H°. Then $\beta/2 = \bar\alpha$. Following [St, Theorem 8.2], we may normalize the homomorphism x_β such that $[x_\alpha(y), x_{\tau\alpha}(y')] = x_\beta(yy')$, where $[a,b] := a^{-1}b^{-1}ab$. We stress that x_β depends on the choice of ordering $(\alpha, \tau\alpha)$ of the set $\{\alpha, \tau\alpha\}$. It is proved in [St, Thm. 8.2(2)] that there are nontrivial elements in U^τ of the form $x_\alpha(y_\alpha)x_{\tau\alpha}(y_{\tau\alpha})x_\beta(y_\beta)$ only if $c_\alpha c_{\tau\alpha} = \pm 1$. In fact we will always have $c_\alpha c_{\tau\alpha} = 1$: since we are dealing with root subgroups we may assume G is adjoint and simple of type A_{2n}, and that τ is the unique (order two) element in $\mathrm{Aut}(\mathrm{PGL}_{2n+1})$ which fixes the standard splitting and induces the order two diagram automorphism; then for all $y \in k$, $x_\alpha(y) = \tau^2 x_{\tau^2\alpha}(y) = \tau x_{\tau\alpha}(c_\alpha y) = x_\alpha(c_{\tau\alpha}c_\alpha y)$.

Assume $\mathrm{char}(k) \neq 2$. As $c_\alpha c_{\tau\alpha} = 1$ is automatic, according to the proof of [St, Thm.8.2(2)] we may define $x^{H_1}_{\bar\alpha} : \mathbb{G}_a \to H_1$ by

$$x^{H_1}_{\bar\alpha}(y) = x_\alpha(y)\, x_{\tau\alpha}(c_\alpha y)\, x_\beta(-c_\alpha y^2/2). \tag{4.7}$$

Let $U_{\bar\alpha} \cong \mathbb{G}_a$ be the image of $x^{H_1}_{\bar\alpha}$. If $\alpha \in \Delta(T,B)$ then $c_\alpha = c_{\tau\alpha} = 1$, and $x^{H_1}_{\bar\alpha}$ is given by

$$x^{H_1}_{\bar\alpha}(y) = x_\alpha(y)\, x_{\tau\alpha}(y)\, x_\beta(-y^2/2). \tag{4.8}$$

Assume $\mathrm{char}(k) = 2$. Then following [St, Thm. 8.2(2)], for $\alpha \in \Psi^+$ representing S_α, y_α and $y_{\tau\alpha}$ are forced to be trivial, and y_β ranges freely, so that we may define $x^{H_1}_{2\bar\alpha} : \mathbb{G}_a \to H_1$ by

$$x^{H_1}_{2\bar\alpha}(y) = x_\beta(y). \tag{4.9}$$

For $i \in \{1,2\}$, in all cases define $U_{i\bar\alpha}$ to be the image of $x^{H_1}_{i\bar\alpha}$ when $x^{H_1}_{i\bar\alpha}$ can be defined; otherwise set $U_{i\bar\alpha} = \{1\}$. Then [St, Theorem 8.2] shows that U^τ is the product of the subgroups $U_{\bar\alpha}$ (or sometimes $U_{2\bar\alpha}$) corresponding to the various S_α's. In particular U^τ is connected. Further, $U_{\bar\alpha}$ (resp. $U_{2\bar\alpha}$) is isomorphic to $\mathbb{G}_a$ whenever $\alpha \in \Delta(T,B)$. This holds for more general $\alpha \in \Psi^+$ too, except possibly when S_α has type 1: *a priori* $U_{\bar\alpha}$ could be trivial if $\alpha \notin \Delta(T,B)$ (but see Remark 4.4). Thus property (iii) holds for $I = \langle\tau\rangle$.

Now we consider (i). The argument of the *Lifting step* above used only Lemma 4.2 and property (iii), and so can be used here to show that $N^\tau \cap H^{\tau,\circ} \twoheadrightarrow W^\tau$. Let $R \subset H^{\tau,\circ}$ be the unipotent radical. By [St, Cor. 7.4], R is contained in a τ-stable Borel subgroup of H°, which we may assume to be B; hence $R \subset U^\tau$. But by the surjectivity of $N^\tau \cap H^{\tau,\circ} \to W^\tau$

and by (iii, iv), only the trivial subgroup of U^τ can be normalized by $N^\tau \cap H^{\tau,\circ}$. Hence $R = 1$ and $H^{\tau,\circ}$ is reductive.

Since U^τ is connected it follows that $B^{\tau,\circ} = T^{\tau,\circ} \cdot U^\tau$. Also, $H^{\tau,\circ}/B^{\tau,\circ}$ is proper, so $B^{\tau,\circ}$ is a parabolic subgroup of $H^{\tau,\circ}$. Thus $B^{\tau,\circ}$ is a Borel subgroup of $H^{\tau,\circ}$, being a connected solvable parabolic subgroup of a reductive group. It follows that $T^{\tau,\circ}$ is a maximal torus of $H^{\tau,\circ}$.

Finally, we need to construct the splitting X^τ. If $\mathrm{char}(k) \neq 2$, then the definition of $x^{H_1}_{\bar{\alpha}}$ above shows that the simple roots for $\Psi(H^{\tau,\circ}, T^{\tau,\circ})$ are the averages $\bar{\alpha}$ of the τ-orbits of the $\alpha \in \Delta(T, B)$. For a simple root $\alpha' \in \Delta(T^{\tau,\circ}, B^{\tau,\circ})$, let

$$X_{\alpha'} := \sum_{\substack{\alpha \in \Delta(T,B) \\ \bar{\alpha} = \alpha'}} X_\alpha.$$

One can check by taking differentials of (4.6) and (4.8) that $X_{\alpha'} \in \mathrm{Lie}(H^{\tau,\circ})_{\alpha'}$, and so $X = \sum_{\alpha'} X_{\alpha'}$ gives the desired splitting. If $\mathrm{char}(k) = 2$, we have to be more careful: $\mathrm{Lie}(H^{\tau,\circ})$ can be smaller than $\mathrm{Lie}(H^\circ)^\tau$, and in fact when S_α is Type 2, $X_\alpha + X_{\tau\alpha}$ will not belong to $\mathrm{Lie}(H^{\tau,\circ})$. Nevertheless, we can define X^τ to be the splitting corresponding to the collection of root-group homomorphisms

$$\{x^{H_1}_{\bar{\alpha}}(y)\} \bigcup \{x^{H_1}_{2\bar{\alpha}}(y)\} \tag{4.10}$$

where the first (resp. second) collection in the union is indexed by the Type 1 (resp. Type 2) subsets S_α (for $\alpha \in \Delta(T, B)$).

It remains to prove (ii) when $I = \langle\tau\rangle$. First assume $\mathrm{char}(k) \neq 2$. Then $(\Psi^\tau)_{\mathrm{red}}$ is a reduced root system with simple roots $\Delta^\tau = \{\bar{\alpha} \mid \alpha \in \Delta\}$. But $\Psi(H^{\tau,\circ}, T^{\tau,\circ})$ is also a reduced root system and we saw in the proof of (i,iii) above that it also has Δ^τ as its set of simple roots. Hence $(\Psi^\tau)_{\mathrm{red}} = \Psi(H^{\tau,\circ}, T^{\tau,\circ})$. Next assume $\mathrm{char}(k) = 2$. Now $(\Psi^\tau)^{\mathrm{red}}$ is a reduced root system with simple roots $\{\bar{\alpha}\} \cup \{2\bar{\alpha}\}$ where the first (resp. second) collection in the union is indexed by Type 1 (resp. Type 2) subsets S_α (for $\alpha \in \Delta$). We saw above that this is precisely the set of simple roots for $\Psi(H^{\tau,\circ}, T^{\tau,\circ})$; hence $(\Psi^\tau)^{\mathrm{red}} = \Psi(H^{\tau,\circ}, T^{\tau,\circ})$. In particular, as $\bar{\alpha} \in (\Psi^\tau)^{\mathrm{red}}$ whenever $\alpha \in \Psi^+$ represents a Type 1 S_α, we now see that $U_{\bar{\alpha}} \cong \mathbb{G}_a$ for such α's (cf. Remark 4.4).

Thus we have proved (i-iii) hold when $I = \langle\tau\rangle$. Note again that when $|\tau|$ is odd Ψ^τ is reduced.

Now suppose $I = S_3$, which will arise in the same way as in the proof of Lemma 4.2. Write $I = \langle\tau_1, \tau_2\rangle$, where τ_1 generates the normal subgroup of order 3, and τ_2 is of order 2. Then by applying the above argument first with $\tau = \tau_1$ and then with $\tau = \tau_2$ (note that τ_2 fixes the splitting $X^{\tau_1} = X$), we see that (i-iii) also hold in this case.

Now consider the most general case, where I is arbitrary. Let Z denote the center of H°. We have a short exact sequence

$$1 \to Z^I \to (H^\circ)^I \to (H^\circ_{\mathrm{ad}})^I \to H^1(I, Z). \tag{4.11}$$

As Z has finite $|I|$-torsion, we see $H^1(I, Z)$ is finite and thus $[(H^\circ)^I]^\circ$ surjects onto $[(H^\circ_{\mathrm{ad}})^I]^\circ$. So we have an exact sequence for H

$$1 \to Z^I \cap H^{I,\circ} \to H^{I,\circ} \to (H^\circ_{\mathrm{ad}})^{I,\circ} \to 1. \tag{4.12}$$

This exact sequence shows that $H^{I,\circ}$ is reductive if $(H^\circ_{\mathrm{ad}})^{I,\circ}$ is reductive. Similarly, (i-iii) for H° follow formally from (i-iii) for H°_{ad}.

Thus, we are reduced to assuming $H = H^\circ_{\mathrm{ad}}$. Then H is a product of simple groups, which are permuted by I and which each carry an action by the stabilizer subgroup of I. We may therefore assume H is simple, and the classification shows we may assume $I = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, or S_3. Each of these cases was handled above, and we conclude that (i-iii) indeed hold for adjoint groups. □

Remark 4.5. Some of Proposition 4.1 appears in [Ri, Lemma A.1], but with the unnecessary assumption that the order $|I|$ is prime to $\mathrm{char}(k)$. The latter assumption is indeed necessary to prove that H^I is reductive, when I is finite but is not assumed to fix any Borel pair (T, B) (see [PY, Thm. 2.1, Rem. 3.5], on which [Ri] relies).

Lemma 4.6. *Assume H, B, T, X, and I are as in Proposition 4.1. Let Z denote the center of H°.*

(i) *Suppose $(H^\circ_{\mathrm{ad}})^I$ is connected. Then the natural map $Z^I \to \pi_0\,(H^\circ)^I$ is surjective.*
(ii) *Let T_{ad} denote the image of T in $(H^\circ)_{\mathrm{ad}}$. If $(T_{\mathrm{ad}})^I$ is connected, then $Z^I(H^\circ)^{I,\circ} = (H^\circ)^I$.*

Proof. For part (i), the key point is that $H^{I,\circ}$ surjects onto $(H^\circ_{\mathrm{ad}})^I$ by (4.12). Part (ii) follows immediately from (i) and Proposition 4.1(d). □

Lemma 4.7. *Suppose H, B, T are as in Proposition 4.1, and assume $I = \langle\tau\rangle$ fixes B, T but not necessarily X. Also assume $\mathrm{char}(k) = 0$ and that H is connected. In the group $H \rtimes \langle\tau\rangle$, the set $(H \rtimes \tau)_{\mathrm{ss}}$ of semisimple elements in the coset $H \rtimes \tau$ is the set of H-conjugates of $T \rtimes \tau$.*

Proof. Conjugates of elements in $T \rtimes \tau$, are semi-simple since τ has finite order and $\mathrm{char}(k) = 0$. Conversely, suppose $h\tau$ is semi-simple. Since H is connected, [St, 7.3] ensures that some τ-conjugate of h lies in B; hence we may assume $h \in B$. Now by [St, Theorem 7.5] applied to the (disconnected) group $B \rtimes \langle\tau\rangle$, $\mathrm{Int}(h\tau)$ fixes a maximal torus $T' \subset B$. Write $T' = bTb^{-1}$ for some $b \in B$. We obtain $b^{-1}h\tau(b) \in N \cap B = T$, thus $h\tau$ is H-conjugate to $T \rtimes \tau$. □

Remark 4.8. For applications in the rest of this article, we only need the case $k = \mathbb{C}$ of these results, and so strictly for the present purposes the above exposition could have been shortened somewhat. However, eventually one hopes to develop a geometric Satake isomorphism for general groups *and* for coefficients in arbitrary fields k (or even in arbitrary commutative rings; see [MV]) and in those more general situations the above results will be needed.

We close this section with a lemma that will be needed for the proof of Theorem 1.3 to be given in §11. We keep the notation and hypotheses of Proposition 4.1, except we assume $\mathrm{char}(k) = 0$ and $I = \langle\tau\rangle$. Fix $\lambda \in k^\times$ which is *not a root of unity.* As in the introduction, let $\mathcal{P}^\tau_{\mathrm{reg}}(H)$ denote the set of pairs $(h \rtimes \tau, e) \in [H \rtimes \tau]_{\mathrm{ss}} \times \mathcal{N}(H)$ with (i) $\mathrm{Ad}(h \rtimes \tau)(e) = \lambda e$, (ii) $\tau(e) = e$, and (iii) e is a *principal* nilpotent element in $\mathrm{Lie}(H^\circ)$. Note that H acts on $\mathcal{P}^\tau_{\mathrm{reg}}(H)$ by the formula $g_1 \cdot (g \rtimes \tau, e) = (g_1 g \tau(g_1)^{-1} \rtimes \tau, \mathrm{Ad}(g_1)(e))$.

Suppose $(g \rtimes \tau, e) \in \mathcal{P}^\tau_{\mathrm{reg}}(H)$. Associated to $e \in \mathrm{Lie}(H)$ is a uniquely determined Borel subgroup $B_e \subset H$, defined as follows (see [Ca, §5.7]). By the Jacobson-Morozov theorem, e belongs to an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. This gives a copy of $\mathfrak{sl}_2$ inside $\mathrm{Lie}(H)$, well-defined up to $C_{H^\circ}(e)$-conjugacy. We can lift the Lie-algebra embedding $\mathfrak{sl}_2 \hookrightarrow \mathrm{Lie}(H)$ to a group embedding $\mathrm{SL}_2 \hookrightarrow H^\circ$, also well-defined up to $C_{H^\circ}(e)$-conjugacy. Consider the cocharacter γ given by $\lambda \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ (viewed inside H°). Let T be a maximal torus of H° containing $\gamma(k^\times)$. Define

$$P_e = \langle T, U_\alpha; \langle \alpha, \gamma\rangle \geq 0\rangle.$$

Then [Ca, Prop. 5.7.1] shows that P_e is a well-defined parabolic subgroup of H° determined by e, and that $C_{H^\circ}(e) \subseteq P_e$.[3] In our case e and hence the semisimple element h is regular, meaning γ is also regular. This implies that P_e is a Borel subgroup, which we henceforth denote by B_e. Note that B_e is preserved by τ. Let U_e denote the unipotent radical of B_e.

Lemma 4.9. *Assume* $\mathrm{char}(k) = 0$*,* $I = \langle\tau\rangle$*, and* $\lambda \in k^\times$ *is not a root of unity. Assume further that* $H = Z(H)H^\circ$*. Let* $(g \rtimes \tau, e) \in \mathcal{P}^\tau_{\mathrm{reg}}(H)$*, and suppose* $T \subset B_e$ *is a maximal torus containing* $\gamma(k^\times)$ *as above, so that we can write* $e = \sum_{\alpha \in \Delta_e} X_\alpha$*, where* $\Delta_e := \Delta(T, B_e)$*. Let* $\delta \in T$ *be an element such that* $\mathrm{Ad}(\delta)(e) = \lambda e$*. Then the* U_e*-orbit of* $(g \rtimes \tau, e)$ *contains an element of the form* $(\delta z \rtimes \tau, e)$*, where* $z \in Z(H)$*.*

Proof. First note that $\mathrm{Ad}(g \rtimes \tau)(e) = \lambda e$ implies $\mathrm{Ad}(g)(e) = \lambda e$, since τ fixes e. Since $H = Z(H)H^\circ$, we may assume $g \in H^\circ$. We have $\delta^{-1} g \in C_{H^\circ}(e) \subseteq B_e$. Hence $g \in B_e$. As in the proof of Lemma 4.7, we may find $u \in U_e$ and $t \in T$ such that $g = u^{-1} t \tau(u)$.

Recall $e = \sum_{\alpha \in \Delta_e} X_\alpha$ where $X_\alpha \neq 0, \forall \alpha$. Let β range over B_e-positive roots with $\mathrm{ht}(\beta) \geq 2$, and for each choose $X_\beta \in \mathrm{Lie}(H^\circ)_\beta \backslash \{0\}$. We can write

$$\mathrm{Ad}(u)(e) = e + \sum_{\mathrm{ht}(\beta) \geq 2} a_\beta X_\beta \tag{4.13}$$

$$\mathrm{Ad}(\tau(u))(e) = e + \sum_{\mathrm{ht}(\beta) \geq 2} a_\beta \, \tau(X_\beta) \tag{4.14}$$

for certain scalars $a_\beta \in k$.

Applying $\mathrm{Ad}(t)$ to (4.14), we get

$$\mathrm{Ad}(t\tau(u))(e) = \mathrm{Ad}(t)(e) + \sum_{\mathrm{ht}(\beta) \geq 2} a_\beta \, (\tau^{-1}\beta)(t) \, \tau(X_\beta).$$

[3] Strictly speaking, [Ca, Prop. 5.7.1] pertains only to the adjoint group $(H^\circ)_{\mathrm{ad}}$, but the same proof applies.

Applying $\mathrm{Ad}(u^{-1})$ to this and using the hypothesis on g yields

$$\lambda e = \mathrm{Ad}(u^{-1}t\tau(u))(e) = \mathrm{Ad}(t)(e) + \sum_{\mathrm{ht}(\beta)\geq 2} Y_\beta$$

for certain $Y_\beta \in \mathrm{Lie}(H^\circ)_\beta$. So $Y_\beta = 0, \forall\beta$ and

$$\mathrm{Ad}(t)(e) = \lambda e. \tag{4.15}$$

As $\delta^{-1}t \in T$ and fixes e under the adjoint action, we obtain $t = \delta \cdot z$ for some $z \in Z(H^\circ)$.

Our hypothesis on g means that applying $\mathrm{Ad}(t)$ to (4.14) yields λ times (4.13). But using (4.15), we see that $\mathrm{Ad}(t)(X_\alpha) = \lambda X_\alpha$, $\forall\alpha \in \Delta_e$, and thus we get

$$\lambda e + \sum_{\mathrm{ht}(\beta)\geq 2} \lambda^{\mathrm{ht}(\beta)} a_\beta \tau(X_\beta) = \lambda e + \sum_{\mathrm{ht}(\beta)\geq 2} \lambda a_\beta X_\beta.$$

Here we used that τ preserves heights. In fact, if we fix a height $h \geq 2$, and let $[\tau]$ denote the matrix given by τ and the basis $\{X_\beta\}_{\mathrm{ht}(\beta)=h}$, we get an equality of column vectors of the form

$$[\tau][\lambda^h a_\beta]_\beta = [\lambda a_\beta]_\beta.$$

Choose $N \geq 1$ with $[\tau]^N = \mathrm{id}$. This entails $a_\beta \lambda^{N(h-1)} = a_\beta$. Since λ is not a root of unity, we deduce that $a_\beta = 0$ for all β.

Thus $\mathrm{Ad}(u)(e) = e$, and $u \cdot (g \rtimes \tau, e) = (\delta z \rtimes \tau, e)$, as desired. □

5. The parameter space

Assume G is a *quasi-split* connected reductive group over F, and fix an F-rational maximal torus/Borel subgroup $T \subset B$ thereof. Let $\widehat{G}$ be the complex dual group of G. By definition, it carries an action by the absolute Galois group Γ over F, which factors through a finite quotient and fixes a splitting of the form $(\widehat{B}, \widehat{T}, \widehat{X})$ (cf. [Ko84, 1.5]), where we may assume $\widehat{T}$ is the complex dual torus for T. Note that since G is quasi-split with Γ-fixed pair (B, T), the Γ-action on $\widehat{T}$ inherited from $\widehat{G}$ agrees with that derived from the Γ-action on $X_*(T) = X^*(\widehat{T})$.[4] We shall use this remark below, applied to the torus T_{sc}.

The group $\widehat{G}^{I_F}$ is reductive by Proposition 4.1 and carries an action by $\tau = \Phi$ which fixes the splitting $(\widehat{T}^{I_F,\circ}, \widehat{B}^{I_F,\circ}, \widehat{X})$ (see Proposition 4.1(a)). Write I for I_F in what follows.

By Lemma 4.7 applied with $H = \widehat{G}^{I,\circ}$, we have a surjection

$$\widehat{T}^{I,\circ} \to [\widehat{G}^{I,\circ} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I,\circ}.$$

Let $\widehat{Z} = Z(\widehat{G})$. Since the torus T_{sc} in G_{sc} is I-induced, its dual torus $\widehat{T}_{\mathrm{ad}}$ is also I-induced, and so $(\widehat{T}_{\mathrm{ad}})^I$ is connected. Then from Lemma 4.6 (ii) with $H = \widehat{G}$, we see $\widehat{Z}^I \cdot \widehat{G}^{I,\circ} = \widehat{G}^I$.

Now multiplying on the left by $\widehat{Z}^I$, the above surjection gives rise to a surjection

$$(\widehat{T}^I)_\Phi \to [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^I. \tag{5.1}$$

[4] See the even more pedantic discussion of dual groups in the second paragraph of section 8.

Let $\widehat{N} = N(\widehat{G}, \widehat{T})$ and $\widehat{W} = W(\widehat{G}, \widehat{T})$. By Proposition 4.1(c), $\widehat{W}^I = W(\widehat{G}^{I,\circ}, \widehat{T}^{I,\circ})$, and $(\widehat{W}^I)^\Phi = W(\widehat{G}^{\Gamma,\circ}, \widehat{T}^{\Gamma,\circ})$. Further, we see that $(\widehat{N}^I)^\Phi = \widehat{N}^\Gamma$ surjects onto $(\widehat{W}^I)^\Phi = \widehat{W}^\Gamma$. We thus have a well-defined surjective map

$$(\widehat{T}^I)_\Phi / \widehat{W}^\Gamma \to [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}} / \widehat{G}^I. \tag{5.2}$$

Proposition 5.1. *The map (5.2) is bijective.*

Proof. It remains to prove the injectivity, which is similar to [Mis, Prop. 11]. Suppose there exist $s, t \in \widehat{T}^I$ and $zg_0 \in Z^I \cdot \widehat{G}^{I,\circ}$ with $(zg_0)^{-1} s \Phi(zg_0) = t$. Write U for the unipotent radical of $\widehat{B}$. Via the Bruhat decompsition write $g_0 = u_0 n_0 v_0$ where $n_0 \in \widehat{N}^I \cap \widehat{G}^{I,\circ}$, $v_0 \in U^I$, and $u_0 \in U^I \cap {}^{n_0}\overline{U}^I$. We have

$$s\Phi(u_0)\Phi(zn_0)\Phi(v_0) = u_0(zn_0)v_0 t,$$

and thus

$$(s\Phi(u_0)s^{-1}) \cdot s\Phi(zn_0) \cdot \Phi(v_0) = u_0 \cdot (zn_0 t) \cdot (t^{-1} v_0 t).$$

Uniqueness of the decomposition yields

$$s\Phi(zn_0) = zn_0 t.$$

The image of $zn_0 \in \widehat{N}^I$ in $\widehat{W}^I$ is therefore Φ-fixed, so lifts (cf. Prop. 4.1) to some element $n_1 \in \widehat{N}^\Gamma$; write $zn_0 = t_1 n_1$ for some $t_1 \in \widehat{T}^I$. The resulting equation

$$n_1^{-1}(t_1^{-1} s \Phi(t_1)) n_1 = t$$

shows that s and t have the same image in $(\widehat{T}^I)_\Phi / \widehat{W}^\Gamma$. □

Corollary 5.2. *If G/F is quasi-split, the set $[\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}} / \widehat{G}^{I_F}$ has the structure of an affine algebraic variety canonically isomorphic to $(\widehat{T}^{I_F})_\Phi / \widehat{W}^\Gamma$.*

Remark 5.3. If G is not quasi-split over F, then as in §8, we consider it as an inner form (G, Ψ) of a group G^* which is quasi-split over F. Then Ψ induces a canonical Γ-isomorphism of based root systems $\psi : \Psi_0(G) \xrightarrow{\sim} \Psi_0(G^*)$. Following [Ko84, §1], recall that a dual group for G^* is a pair $(\widehat{G^*}, \iota)$, where $\widehat{G^*}$ is a connected reductive group over $\mathbb{C}$, where $\iota : \Psi_0(G^*)^\vee \xrightarrow{\sim} \Psi_0(\widehat{G^*})$ is an Γ^*-isomorphism of based root systems, and where Γ^* fixes some splitting for $\widehat{G^*}$.[5] If $(\widehat{G^*}, \iota)$ is a dual group for G^*, then $(\widehat{G^*}, \iota \circ \check{\psi}^{-1})$ is a dual group for G. Thus (G^*, Ψ) gives rise to canonical identifications ${}^L G^* = {}^L G$ and

$$[\widehat{G^*}^{I_F^*} \rtimes \Phi^*]_{\mathrm{ss}} / \widehat{G^*}^{I_F^*} = [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}} / \widehat{G}^{I_F}. \tag{5.3}$$

Thus the right hand side inherits the structure of an affine algebraic variety from the left hand side.

[5] We write Γ^*, I_F^*, etc., to indicate Galois actions on G^*.

6. Construction of parameters: quasi-split case

Now again assume G/F is quasi-split. Let A be a maximal F-split torus in G, and suppose $T = \mathrm{Cent}_G(A)$; let $W = W(G,T)$ and recall that since G is quasi-split, W^Γ is the relative Weyl group $W(G,A)$. There is a Γ-equivariant isomorphism $W \cong \widehat{W}$. Putting this together with Proposition 5.1 yields the following result.

Proposition 6.1. *Assume G is quasi-split over F. There is a natural bijection*

$$(\widehat{T}^{I_F})_\Phi/W(G,A) \xrightarrow{\sim} [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}. \tag{6.1}$$

Let $J \subset G(F)$ be any parahoric subgroup, and let $\pi \in \Pi(G/F, J)$. By [H13, §11.5], there exists a weakly unramified character $\chi \in (\widehat{T}^{I_F})_\Phi/W(G,A)$ such that π is an irreducible subquotient of the normalized induction $i_B^G(\chi)$.

Definition 6.2. Define $s(\pi) \in S(G) := [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$ to be the image of $\chi \in (\widehat{T}^{I_F})_\Phi/W(G,A)$ under the bijection (6.1).[6]

We have the Bernstein isomorphism of [H13, 11.10.1][7]

$$S : \mathcal{Z}(G(F), J) \xrightarrow{\sim} \mathbb{C}[(\widehat{T}^{I_F})_\Phi/W(G,A)]. \tag{6.2}$$

By [H13, §11.8], $z \in \mathcal{Z}(G(F), J)$ acts on $i_B^G(\chi)^J$ by the scalar $S(z)(\chi)$. Then we have the following characterization of $s(\pi)$: *any element $z \in \mathcal{Z}(G(F), J)$ acts on π^J by the scalar $S(z)(s(\pi))$.*

Lemma 6.3. *The map $\pi \mapsto s(\pi)$ is compatible with change of level $J' \subset J$.*

Proof. Clearly $\Pi(G/F, J) \subset \Pi(G/F, J')$, and the compatibility simply reduces to the compatibility between Bernstein isomorphisms when $J' \subset J$. The latter follows from the construction of [H13, 11.10.1]. $\square$

7. Second parametrization of K-spherical representations: quasi-split case

Continue to assume G is quasi-split over F, but take $J = K$ to be a maximal special parahoric subgroup. In this case the Satake parameter can be described in another way.

Theorem 7.1. *Assume G, K as above. We have the following parametrization of* $\Pi(G/F, K)$

$$\Pi(G/F, K) \xrightarrow[\sim]{(3.1)} (\widehat{T}^{I_F})_\Phi/W(G,A) \xrightarrow[\sim]{(6.1)} [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}.$$

We can also realize the Satake parameter $s(\pi)$ for $\pi \in \Pi(G/F, K)$ to be the image of π under this map. This image $s(\pi)$ may be characterized as follows: it is the unique element of the affine variety $[\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$ such that

$$\mathrm{tr}(f|\pi) = S(f)(s(\pi))$$

[6] For the independence of the map $\pi \mapsto s(\pi)$ from auxiliary choices such as A, see the discussion of (9.1).

[7] We use the letter S for this map, because when $J = K$ it is just the Satake isomorphism (3.2).

where $S(f)$ is the Satake transform for any $f \in \mathcal{H}(G(F), K)$. (The Satake isomorphism (3.2) is just a specific instance of a Bernstein isomorphism and Lemma 6.3 shows the two ways of constructing $s(\pi)$ coincide.)

8. Review of transfer homomorphisms

In order to define Satake parameters for general groups, we need to recall the normalized transfer homomorphisms introduced in [H13, §11]. Let G^* be a quasi-split group over F. Let F^s denote a separable closure of F, and set $\Gamma = \mathrm{Gal}(F^s/F)$. Recall that an inner form of G^* is a pair (G, Ψ) consisting of a connected reductive F-group G and a Γ-stable $G^*_{\mathrm{ad}}(F^s)$-orbit Ψ of F^s-isomorphisms $\psi : G \to G^*$. The set of isomorphism classes of pairs (G, Ψ) corresponds bijectively to $H^1(F, G^*_{\mathrm{ad}})$. As before, we will write Γ^*, I^*_F, etc., to indicate Galois actions on G^*.

In the construction of transfer homomorphisms, we start with the choice of some primary data: A, A^*, and $\widehat{B^*} \supset \widehat{T^*}$. Here, A (resp. A^*) is a maximal F-split torus in G (resp. G^*). We will set $M = \mathrm{Cent}_G(A)$ and $T^* = \mathrm{Cent}_{G^*}(A^*)$, a maximal torus in G^*. The Borel/torus pair $\widehat{B^*} \supset \widehat{T^*}$ in $\widehat{G^*}$ is specified as follows: we require $\widehat{T^*}, \widehat{B^*}$ to be part of some Γ^*-fixed splitting $(\widehat{T^*}, \widehat{B^*}, \widehat{X^*})$ (see Remark 5.3). Let ι be as in Remark 5.3. Since G^* is quasi-split, ι induces a Γ^*-isomorphism $X_*(T^*) \xrightarrow{\sim} X^*(\widehat{T^*})$.

Now we make some secondary choices: choose an F-parabolic subgroup $P \subset G$ having M as Levi factor, and an F-rational Borel subgroup $B^* \subset G^*$ having T^* as Levi factor. Then there exists a unique parabolic subgroup $P^* \subset G^*$ such that $P^* \supseteq B^*$ and P^* is $G^*(F^s)$-conjugate to $\psi(P)$ for every $\psi \in \Psi$. Let M^* be the unique Levi factor of P^* containing T^*. Then define

$$\Psi_M = \{\psi \in \Psi \mid \psi(P) = P^*,\ \psi(M) = M^*\}.$$

(Note we suppress the dependence of Ψ_M on P, B^*.) The set Ψ_M is a nonempty Γ-stable $M^*_{\mathrm{ad}}(F^s)$-orbit of F^s-isomorphisms $M \to M^*$, and so (M, Ψ_M) is an inner form of M^*. Choose any $\psi_0 \in \Psi_M$. Then since $\psi_0|A$ is F-rational, $\psi_0(A)$ is an F-split torus in $Z(M^*)$ and hence $\psi_0(A) \subseteq A^*$.

The F-Levi subgroup M^* corresponds to a Γ^*-invariant subset Δ_{M^*} of the B^*-positive simple roots $\widehat{\Delta} \subset X^*(T^*)$. It follows that $\iota(\Delta^\vee_{M^*})$ is a set $\Delta_{\widehat{M^*}}$ of $\widehat{B^*}$-positive simple roots in $X^*(\widehat{T^*})$ for some uniquely determined Γ^*-stable Levi subgroup $\widehat{M^*} \supset \widehat{T^*}$. Note that ι defines a Γ^*-isomorphism $\iota : \Psi_0(M^*)^\vee \xrightarrow{\sim} \Psi_0(\widehat{M^*})$. Writing $\widehat{X^*} = \{\widehat{X^*}_\alpha\}_{\alpha \in \widehat{\Delta}}$, we see that $\widehat{M^*}$ has a Γ^*-fixed splitting $(\widehat{T^*}, \widehat{B^*}_{\widehat{M^*}}, \{\widehat{X^*}_\alpha\}_{\alpha \in \Delta_{\widehat{M^*}}})$, where $\widehat{B^*}_{\widehat{M^*}} := \widehat{B^*} \cap \widehat{M^*}$. Hence $(\widehat{M^*}, \iota)$ is a dual group for M^*.

Thus, for every $\psi_0 \in \Psi_M$, we have a Γ-equivariant homomorphism

$$\hat{\psi}_0 : Z(\widehat{M}) \xrightarrow{\sim} Z(\widehat{M^*}) \hookrightarrow \widehat{T^*}.$$

(See Remark 5.3.)

We obtain a morphism of affine algebraic varieties

$$\begin{aligned} t^{\widehat{T^*},\widehat{B^*}}_{A^*,A} : (Z(\widehat{M})^{I_F})_{\Phi_F}/W(G,A) &\longrightarrow (\widehat{T^*}^{I^*_F})_{\Phi^*_F}/W(G^*,A^*) \\ \hat{m} &\longmapsto \hat{\psi}_0(\hat{m}). \end{aligned}$$

The morphism $t^{\widehat{T^*},\widehat{B^*}}_{A^*,A}$ is independent of the choices of P and B^*. Henceforth we will follow the notation of [HRo, §12.2] and [H13, §11], by writing $t_{A^*,A}$ instead of $t^{\widehat{T^*},\widehat{B^*}}_{A^*,A}$.

We now recall the definition of a normalized version of $t_{A^*,A}$, for which we need to refine the choice of $\psi_0 \in \Psi_M$ somewhat. Following [H13, Lemma 11.12.4], given the choice of $P \supset M$ and $B^* \supset T^*$ used to define Ψ_M, choose any F^{un}-*rational* $\psi_0 \in \Psi_M$ and define a morphism of affine algebraic varieties

$$\begin{aligned} \tilde{t}_{A^*,A} : (Z(\widehat{M})^{I_F})_{\Phi_F}/W(G,A) &\longrightarrow (\widehat{T^*}^{I^*_F})_{\Phi^*_F}/W(G^*,A^*) \\ \hat{m} &\longmapsto \delta_{B^*}^{-1/2} \cdot \hat{\psi}_0(\delta_P^{1/2}\hat{m}). \end{aligned} \tag{8.1}$$

This makes sense as δ_P (resp. δ_{B^*}) is a weakly unramified character of $M(F)$ (resp. $T^*(F)$), and so can be regarded as an element of $(Z(\widehat{M})^{I_F})_{\Phi_F}$ (resp. $(\widehat{T^*}^{I^*_F})_{\Phi^*_F}$), by [H13, (3.3.2)].

Lemma 8.1. *The morphism $\tilde{t}_{A^*,A}$ is well-defined and independent of the choice of P, B^*, and F^{un}-rational $\psi_0 \in \Psi_M$ used in its construction.*

Proof. The independence statement and the compatibility with the Weyl group actions are proved in [H13, 11.12.4]. □

Lemma 8.2. *The morphism (8.1) is a closed immersion.*

Proof. We first prove that the map $(Z(\widehat{M})^{I_F})_{\Phi_F} \to (\widehat{T^*}^{I^*_F})_{\Phi^*_F}$ given by $\hat{m} \mapsto \delta_{B^*}^{-1/2}\hat{\psi}_0(\delta_P^{1/2}\hat{m})$ is a closed immersion. For this it is clearly enough to show that the *unnormalized* map $\hat{m} \mapsto \hat{\psi}_0(\hat{m})$ is a closed immersion. But this follows from the surjectivity of the corresponding map

$$t_{A^*,A} : X^*(\widehat{T^*})^{\Phi^*_F}_{I^*_F} \to X^*(Z(\widehat{M}))^{\Phi_F}_{I_F}, \tag{8.2}$$

which was proved in [H13, Remark 11.12.2]. In fact this is done by interpreting (8.2), via the Kottwitz isomorphism, as the natural map

$$T^*(F)/T^*(F)_1 \longrightarrow M^*(F)/M^*(F)_1 \xrightarrow[\sim]{\psi_0^{-1}} M(F)/M(F)_1. \tag{8.3}$$

We therefore have a surjective normalized variant

$$\tilde{t}_{A^*,A} : T^*(F)/T^*(F)_1 \longrightarrow M^*(F)/M^*(F)_1 \xrightarrow[\sim]{\psi_0^{-1}} M(F)/M(F)_1. \tag{8.4}$$

Now recall that [H13, 11.12.3] constructs a bijective map

$$W(G,A) \xrightarrow[\sim]{\psi_0^{\natural}} W(G^*,A^*)/W(M^*,A^*) \tag{8.5}$$

defined as follows. Let $F^{\rm un}$ be the maximal unramified extension of F in F^s, and let L denote the completion of $F^{\rm un}$. Let S^* be the $F^{\rm un}$-split component of T^*. Choose a maximal $F^{\rm un}$-split torus $S \subset G$ which is defined over F and which contains A, and set $T = {\rm Cent}_G(S)$. Choose $\psi_0 \in \Psi_M$ such that ψ_0 is defined over $F^{\rm un}$ and has $\psi_0(S) = S^*$, hence also $\psi_0(T) = T^*$. Now suppose $w \in W(G, A)$. We may choose a representative $n \in N_G(S)(L)^{\Phi_F}$ (cf. [HRo]). There exists $m^*_n \in N_{M^*}(S^*)(L)$ such that $\psi_0(n)m^*_n \in N_{G^*}(A^*)(F)$. Then define $\psi_0^\natural(w)$ to be the image of $\psi_0(n)m^*_n$ in $W(G^*, A^*)/W(M^*, A^*)$.

Now the desired surjectivity of

$$\mathbb{C}[T^*(F)/T^*(F)_1]^{W(G^*,A^*)} \xrightarrow{\tilde{t}_{A^*A}} \mathbb{C}[M(F)/M(F)_1]^{W(G,A)} \tag{8.6}$$

follows without difficulty using the surjectivity of (8.4) and the isomorphism (8.5), because $W(M^*, A^*)$ and $N_{M^*}(S^*)(L)$ act trivially on $M^*(F)/M^*(F)_1$. Indeed, for $m \in M(F)/M(F)_1$, define $\Sigma_m \in \mathbb{C}[M(F)/M(F)_1]^{W(G,A)}$ by $\Sigma_m := \sum_{w \in W(G,A)} w \cdot m$. Suppose $t^* \in T^*(F)/T^*(F)_1$ maps to m under (8.4), and define $\Sigma_{t^*} \in \mathbb{C}[T^*(F)/T^*(F)_1]^{W(G^*,A^*)}$ by $\Sigma_{t^*} := \sum_{w^* \in W(G^*,A^*)} w^* \cdot t^*$. Then (8.6) sends Σ_{t^*} to $|W(M^*, A^*)| \cdot \Sigma_m$. □

Recall the definition of the normalized transfer homomorphism on the level of Bernstein centers.

Definition 8.3. ([H13, 11.12.1]) Let $J \subset G(F)$ and $J^* \subset G^*(F)$ be any parahoric subgroups and choose maximal F-split tori A resp. A^* to be in good position[8] relative to J resp. J^*. Then we define the *normalized transfer homomorphism* $\tilde{t} : \mathcal{Z}(G^*(F), J^*) \rightarrow \mathcal{Z}(G(F), J)$ to be the unique homomorphism making the following diagram commute

$$\begin{array}{ccc}
\mathcal{Z}(G^*(F), J^*) & \xrightarrow{\quad\tilde{t}\quad} & \mathcal{Z}(G(F), J) \\
\wr\big\downarrow S & & \wr\big\downarrow S \\
\mathbb{C}[X^*(\widehat{T^*})^{\Phi^*_F}_{I^*_F}]^{W(G^*,A^*)} & \xrightarrow{\tilde{t}_{A^*,A}} & \mathbb{C}[X^*(Z(\widehat{M}))^{\Phi_F}_{I_F}]^{W(G,A)}.
\end{array}$$

We use S to denote the Bernstein isomorphisms described in [H13, 11.10.1]. As explained in [H13, Def. 11.12.5], $\tilde{t}$ is independent of the choices for A, A^*, and $\widehat{B^*} \supset \widehat{T^*}$, and is a completely canonical homomorphism.

Corollary 8.4. (*of Lemma 8.2*) *The normalized transfer homomorphism* $\tilde{t} : \mathcal{Z}(G^*(F), J^*) \rightarrow \mathcal{Z}(G(F), J)$ *is* surjective.

We now present an alternative way to characterize the maps $\tilde{t}$, reformulating slightly [H13, 11.12.6].

Proposition 8.5. *Choose* $A, A^*, \psi_0 \in \Psi_M$ *as needed in Definition 8.3. For each subtorus* $A_L \subseteq A$, *let* $L = {\rm Cent}_G(A_L)$ *and* $L^* = \psi_0(L)$, *so that* ψ_0 *restricts to an inner twising*

[8]This means that in the Bruhat-Tits building $\mathcal{B}(G_{\rm ad}, F)$, the facet corresponding to J is contained in the apartment corresponding to A.

$L \to L^*$ of F-Levi subgroups of G resp. G^*. Set $J_L = J \cap L(F)$. Then the family of normalized transfer homomorphisms $\tilde{t} : \mathcal{Z}(L^*(F), J^*_{L^*}) \to \mathcal{Z}(L(F), J_L)$ is the unique family with the following properties:

(a) *The $\tilde{t}$ are compatible with the constant term homomorphisms c^G_L, in the sense that the following diagrams commute for all L:*

$$\begin{tikzcd} \mathcal{Z}(G^*(F), J^*) \arrow[r, "\tilde{t}"] \arrow[d, hook, "c^{G^*}_{L^*}"'] & \mathcal{Z}(G(F), J) \arrow[d, hook, "c^G_L"'] \\ \mathcal{Z}(L^*(F), J^*_{L^*}) \arrow[r, "\tilde{t}"] & \mathcal{Z}(L(F), J_L). \end{tikzcd}$$

(b) *For $L = M$ and $z \in \mathcal{Z}(M^*(F), J^*_{M^*})$, the function $\tilde{t}(z)$ is given by integrating z over the fibers of the Kottwitz homomorphism $\kappa_{M^*}(F)$. (Note $M_{\rm ad}$ is anisotropic over F.)*

The constant term homomorphisms here are defined in [H13, 11.11] as follows: suppose $Q = LR$ is an F-rational parabolic subgroup with Levi factor L and unipotent radical R. Given $z \in \mathcal{Z}(G(F), J)$, define $c^G_L(z) \in \mathcal{Z}(L(F), J_L)$ by

$$c^G_L(z)(l) = \delta_Q^{1/2}(l) \int_{R(F)} z(lr)\, dr = \delta_Q^{-1/2}(l) \int_{R(F)} z(rl)\, dr,$$

for $l \in L(F)$, where ${\rm vol}_{dr}(J \cap R(F)) = 1$. It is proved as in [H09, Lemma 4.7.2] that $c^G_L(z)$ really does belong to the center of $\mathcal{H}(L(F), J_L)$ and is independent of the choice of Q having L as a Levi factor.

9. Construction of parameters: general case

Suppose G is any connected reductive group over F, and $J \subset G(F)$ is a parahoric subgroup. Fix our primary data A, A^* and $\widehat{B^*} \supset \widehat{T^*}$ as in the construction of $\tilde{t}_{A^*,A}$ in (8.1).

Let $\pi \in \Pi(G/F, J)$. By [H13, §11.5], there exists a weakly unramified character $\chi \in (Z(\widehat{M})^{I_F})_\Phi/W(G, A)$ such that π is an irreducible subquotient of the normalized induction $i^G_P(\chi)$.

Definition 9.1. Define $s(\pi) \in [\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$ to be the image of $\chi \in (Z(\widehat{M})^{I_F})_\Phi/W(G, A)$ under the map

(9.1)

$$(Z(\widehat{M})^{I_F})_\Phi/W(G, A) \xhookrightarrow{\tilde{t}_{A^*,A}} (\widehat{T^*}^{I^*_F})_\Phi/W(G^*, A^*) \xrightarrow[\sim]{(6.1)} [\widehat{G^*}^{I^*_F} \rtimes \Phi^*]_{\rm ss}/\widehat{G^*}^{I^*_F} \overset{(5.3)}{=\!=} [\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}.$$

Define $S(G)$ to be the image of this map, which is a closed subvariety of $[\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F}$ by Lemma 8.2.

Let us prove that the set $S(G)$ and the element $s(\pi) \in S(G)$ are independent of the primary choices A, A^*, $\widehat{B^*} \supset \widehat{T^*}$ we have made in their construction. Because $\widehat{G^*}^{\Gamma^*}$ acts transitively on Γ^*-fixed splittings ([Ko84, 1.7]), the map (9.1) is already independent of the pair $\widehat{B^*} \supset \widehat{T^*}$. The independence of the map $\pi \mapsto s(\pi)$ from A (resp. A^*) results from the fact that any other choice for A (resp. A^*) would be $G(F)$-(resp. $G^*(F)$-)conjugate to it.

Now suppose A, J are as above. Then we have the Bernstein isomorphism of [H13, 11.9.1]

$$S : \mathcal{Z}(G(F), J) \xrightarrow{\sim} \mathbb{C}[(Z(\widehat{M})^{I_F})_\Phi / W(G, A)].$$

By [H13, §11.8], $z \in \mathcal{Z}(G(F), J)$ acts on $i_P^G(\chi)^J$ by the scalar $S(z)(\chi)$. Therefore we have the following characterization of $s(\pi)$: choose any parahoric subgroup $J^* \subset G^*(F)$; then for all $z^* \in \mathcal{Z}(G^*(F), J^*)$ we have

$$\mathrm{tr}(\tilde{t}(z^*) \,|\, \pi) = \dim(\pi^J)\, S(z^*)(s(\pi)).$$

In particular, when G is quasi-split, the map $\pi \mapsto s(\pi)$ defined here coincides with the map defined in Definition 6.2. Further, in the general case $\pi \mapsto s(\pi)$ is compatible with change of level $J' \subset J$ in the same sense as in Lemma 6.3.

10. Second parametrization of K-spherical representations: general case

Let $K \subset G(F)$ be a special maximal parahoric subgroup. Putting together the isomorphism (3.1) with the map (9.1), we obtain the following.

Theorem 10.1. *There is a canonical parametrization of* $\Pi(G/F, K)$

$$\Pi(G/F, K) \xrightarrow[\sim]{\pi \mapsto s(\pi)} S(G) \hookrightarrow [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}.$$

Furthermore, $S(G) = [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$ *if and only if* G/F *is quasi-split.*

Proof. The parametrization is immediate, and the "only if" results from the strict inequality $\dim (Z(\widehat{M})^{I_F})_\Phi = \dim (Z(\widehat{M^*})^{I_F^*})_{\Phi^*} < \dim (\widehat{T^*}^{I_F^*})_{\Phi^*}$ if M is not a maximal torus in G. (The inequality follows from Lemma 10.2 below.) This proves items (A) and (B) of Theorem 1.1. $\square$

Lemma 10.2. *For any connected reductive F-group G, the dimension of the diagonalizable group $(Z(\widehat{G})^{I_F})_\Phi$ is the rank of the maximal F-split torus in the center of G.*

Proof. Fix an F-rational maximal torus $T \subset G$ and set $T_{\mathrm{der}} = T \cap G_{\mathrm{der}}$. Define the cocenter torus $D = G/G_{\mathrm{der}} = T/T_{\mathrm{der}}$. The torus $Z(G)^\circ$ is isogenous to D, hence there is a perfect Γ-equivariant $\mathbb{Q}$-valued pairing between $X^*(Z(G)^\circ)_{\mathbb{Q}}$ and $X_*(D)_{\mathbb{Q}}$. A result of Borovoi gives a Γ-equivariant isomorphism $X_*(T)/X_*(T_{\mathrm{sc}}) = X^*(Z(\widehat{G}))$, where T_{sc} denotes the pull-back of T_{der} along the covering $G_{\mathrm{sc}} \to G_{\mathrm{der}}$. It follows that $X_*(D)_{\mathbb{Q}} = X^*(Z(\widehat{G}))_{\mathbb{Q}}$. We obtain a perfect $\mathbb{Q}$-valued pairing between $\big(X^*(Z(G)^\circ)\big)_\Gamma \otimes \mathbb{Q}$ and $\big(X^*(Z(\widehat{G}))\big)^\Gamma \otimes \mathbb{Q} = X^*\big((Z(\widehat{G})^{I_F})_\Phi\big) \otimes \mathbb{Q}$. The lemma follows. $\square$

11. Proof of Theorem 1.3

Thanks to (11.2) below, the equivalence $(i) \Leftrightarrow (iii)$ is fairly obvious from the definition of $S(G)$. We concentrate on $(i) \Leftrightarrow (ii)$. We retain the notation from §8.

Lemma 11.1. *Suppose that the principal nilpotent element $\widehat{X^*}_{\widehat{M^*}} := \sum_{\alpha \in \Delta_{\widehat{M^*}}} \widehat{X^*}_\alpha$ is part of a Γ^*-fixed splitting $(\widehat{T^*}, \widehat{B^*}_{\widehat{M^*}}, \widehat{X^*}_{\widehat{M^*}})$ for $\widehat{M^*}$. Let $\widehat{m^*} \in (Z(\widehat{M^*})^{I^*_F})_{\Phi^*}$. Then*

$$\mathrm{Ad}(\delta^{-1/2}_{B^*_{M^*}} \widehat{m^*} \rtimes \Phi^*)(\widehat{X^*}_{\widehat{M^*}}) = q_F \widehat{X^*}_{\widehat{M^*}}. \tag{11.1}$$

Proof. Thanks to the Kottwitz isomorphism

$$\mathrm{Hom}_{\mathrm{grp}}(T^*(F)/T^*(F)_1, \mathbb{C}^\times) = (\widehat{T^*}^{I^*_F})_{\Phi^*},$$

$\delta^{-1/2}_{B^*_{M^*}}$ is naturally an element of $(\widehat{T^*}^{I^*_F})_{\Phi^*}$. The left hand side of (11.1) is well-defined. Let $\varpi \in F^\times$ be any uniformizer, and let $\rho^*_{\widehat{M^*}}$ be the half-sum of the $\widehat{B^*}_{\widehat{M^*}}$-positive roots in $X^*(\widehat{T^*})$. The homomorphism $X^*(\widehat{T^*}) \to \mathbb{C}^\times$ given by $\lambda \mapsto \delta^{-1/2}_{B^*_{M^*}}(\varpi^\lambda)$ corresponds to an element of $\widehat{T^*}^{I^*_F}$ which projects to $\delta^{-1/2}_{B^*_{M^*}}$ under $\widehat{T^*}^{I^*_F} \to (\widehat{T^*}^{I^*_F})_{\Phi^*}$; denote this element also by $\delta^{-1/2}_{B^*_{M^*}}$. It is clear that $\mathrm{Ad}(\delta^{-1/2}_{B^*_{M^*}})$ acts on $\widehat{X^*}_{\widehat{M^*}}$ by the scalar $\delta^{-1/2}_{B^*_{M^*}}(\varpi^\alpha) = |\varpi^{-\langle \alpha, \rho^*_{\widehat{M^*}}\rangle}|_F = q_F$ (here $\alpha \in \Delta_{\widehat{M^*}}$ is arbitrary). The result follows because $\mathrm{Ad}(\widehat{m^*} \rtimes \Phi^*)$ fixes $\widehat{X^*}_{\widehat{M^*}}$. □

As in §8, we fix $\psi_0 \in \Psi_M$ as needed to define the normalized transfer homomorphism $\tilde{t}_{A^*,A}$. We have the identity

$$\delta^{-1/2}_{B^*_{M^*}} = \delta^{-1/2}_{B^*} \hat{\psi}_0(\delta_P^{1/2}) \tag{11.2}$$

in $\widehat{T^*}^{I^*_F}$. Recall we have a canonical identification $[\widehat{G^*}^{I^*_F} \rtimes \Phi^*]_{\mathrm{ss}}/\widehat{G^*}^{I^*_F} = [\widehat{G}^{I_F} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^{I_F}$. Therefore elements of $S(G)$ can be represented by elements of the form $\delta^{-1/2}_{B^*_{M^*}}\widehat{m^*}$ where $\widehat{m^*}$ is as in Lemma 11.1. Further, by Proposition 4.1(a), $\widehat{M^*}^{I^*_F}$ has a splitting of the form $(\widehat{T^*}^{I^*_F}, \widehat{B^*}^{I^*_F}_{\widehat{M^*}}, \widehat{X^*}_{\widehat{M^*}})$ where $\widehat{X^*}_{\widehat{M^*}}$ is as in Lemma 11.1. That lemma therefore implies that every element of $S(G)$ has the property stated in Theorem 1.3 (ii).

Conversely, suppose $(\hat{g} \rtimes \Phi, x) \in \mathcal{P}^{\Phi}_{\mathrm{reg}}(\widehat{M}^{I_F})$. We want to prove that $\hat{g} \rtimes \Phi$ belongs to $S(G)$. As we may work entirely in $\widehat{M^*}^{I^*_F}$, we may as well assume $M^* = G^*$. Now $x \in \mathcal{N}(\widehat{G^*}^{I^*_F})$ has $\Phi^*(x) = x$, so in fact x is a principal nilpotent in $\mathrm{Lie}(\widehat{G^*}^{\Gamma^*_F})$. By Proposition 4.1(a), there is a splitting $(\widehat{T^*}^{\Gamma^*_F}, \widehat{B^*}^{\Gamma^*_F}, \widehat{X^*})$ for $\widehat{G^*}^{\Gamma^*_F}$, where $\widehat{X^*} = \sum_{\alpha \in \Delta_{\widehat{G^*}}} \widehat{X^*}_\alpha$ comes from a Γ^*_F-fixed splitting $(\widehat{T^*}, \widehat{B^*}, \widehat{X^*})$ for $\widehat{G^*}$. Being a principal nilpotent element in $\mathcal{N}(\widehat{G^*}^{\Gamma^*_F})$, x is $\widehat{G^*}^{\Gamma^*_F}$-conjugate to such an element $\widehat{X^*}$; hence we may assume $x = \widehat{X^*}$. We apply Lemma 4.9 with $H = \widehat{G^*}^{I^*_F}$, $\tau = \Phi^*$, $\lambda = q_F$, $e = \widehat{X^*}$, and $\delta = \delta^{-1/2}_{B^*}$, where (T^*, B^*) corresponds to $(\widehat{T^*}, \widehat{B^*})$. Lemma 4.9 asserts that we may assume $(\hat{g} \rtimes \Phi^*, \widehat{X^*}) = (\delta^{-1/2}_{B^*}\hat{z} \rtimes \Phi^*, \widehat{X^*})$ for some $\hat{z} \in Z(\widehat{G^*}^{I^*_F})$. By Lemma 4.6(ii), $\widehat{G^*}^{I^*_F} = Z(\widehat{G^*})^{I^*_F} \cdot \widehat{G^*}^{I^*_F,\circ}$, so we may write $\hat{z} = \hat{z}_1 \cdot \hat{z}_2$ where $\hat{z}_1 \in Z(\widehat{G^*})^{I^*_F}$ and $\hat{z}_2 \in Z(\widehat{G^*}^{I^*_F,\circ})$. But $\hat{z}_2$ is an element of $\widehat{T^*}^{I^*_F,\circ}$ such that $\mathrm{Ad}(\hat{z}_2)(\widehat{X^*}) = \widehat{X^*}$. Thus in fact $\hat{z}_2$ belongs to $Z(\widehat{G^*})$, and hence $\hat{z}$ belongs to $Z(\widehat{G^*})^{I^*_F}$. This proves that $(\widehat{g} \rtimes \Phi^*, x)$ belongs to $S(G)$. $\square$

12. A transfer map $\Pi(G, K) \to \Pi(G^*, K^*)$

Let $K \subset G(F)$ and $K^* \subset G^*(F)$ be special maximal parahoric subgroups. We shall define an operation

$$\Pi(G/F, K) \hookrightarrow \Pi(G^*/F, K^*)$$
$$\pi \mapsto \pi^*$$

which is dual to $\tilde{t} : \mathcal{H}(G^*(F), K^*) \to \mathcal{H}(G(F), K)$. We identify ${}^LG = {}^LG^*$ as in Remark 5.3. Given π we have $s(\pi) \in S(G) \subseteq [\widehat{G^*}^{I^*_F} \rtimes \Phi^*]_{\rm ss}/\widehat{G^*}^{I^*_F} = S(G^*)$.

Definition 12.1. We define $\pi^* \in \Pi(G^*/F, K^*)$ to be the unique isomorphism class with $s(\pi^*) = s(\pi)$.

Clearly π^* is characterized by the equalities for all $f^* \in \mathcal{H}(G^*(F), K^*)$

$$\mathrm{tr}(\tilde{t}(f^*) \,|\, \pi) = S(\tilde{t}(f^*))(s(\pi)) = S(f^*)(s(\pi^*)) = \mathrm{tr}(f^* \,|\, \pi^*).$$

The middle equality follows from the diagram in Definition 8.3 (taking $J = K$ and $J^* = K^*$). We remark that the character identity directly characterizes π in terms of π^* because $f^* \mapsto \tilde{t}(f^*)$ gives a *surjective* map $\mathcal{H}(G^*(F), K^*) \to \mathcal{H}(G(F), K)$ (Cor. 8.4).

13. Relation with local Langlands correspondence

13.1. **Construction of s-parameter in Deligne-Langlands correspondence.** The Satake parameter $s(\pi)$ should give us part of the local Langlands parameter associated to $\pi \in \Pi(G/F, J)$.

Conjecture 13.1. *Let $W'_F := W_F \rtimes \mathbb{C}$ be the Weil-Deligne group. If $\pi \in \Pi(G/F, J)$ has local Langlands parameter $\varphi_\pi : W'_F \to {}^LG$, then*

$$\varphi_\pi(\Phi) = s(\pi) \tag{13.1}$$

as elements in $[\widehat{G} \rtimes \Phi]_{\rm ss}/\widehat{G}$.

Note that we still denote by $s(\pi)$ its image under the natural map $[\widehat{G}^{I_F} \rtimes \Phi]_{\rm ss}/\widehat{G}^{I_F} \to [\widehat{G} \rtimes \Phi]_{\rm ss}/\widehat{G}$.

Remark 13.2. Put another way, the conjecture predicts the s-parameter in the Deligne-Langlands triple (s, u, ρ) which is hypothetically attached to a representation π with Iwahori-fixed vectors (note that the works of Kazhdan-Lusztig [KL], Lusztig [L1, L2] construct the entire triple unconditionally for many p-adic groups, but not for the most general p-adic groups).

Remark 13.3. This is similar to [Mis, Thm. 2], which discusses the case where G is quasi-split and split over a tamely ramified extension of F.

When G/F is quasi-split, (13.1) is predicted by the compatibility of the local Langlands correspondence (LLC) with normalized parabolic induction, as follows. Recall the property LLC+ ([H13, §5.2]), which is LLC for G and all of its F-Levi subgroups, plus the compatibility of infinitesimal characters (restrictions of Langlands parameters to W_F) with respect to normalized parabolic induction. Write G, T, B, etc. in place of G^*, T^*, B^* etc. from §8. By [H13, §11.5], there is a weakly unramified character χ on $T(F)$ such that π is a subquotient of $i_B^G(\chi)$. Assuming LLC+ holds, we expect $\varphi_\pi|_{W_F} = \varphi_\chi|_{W_F}$, the latter taking values in ${}^LT \hookrightarrow {}^LG$. Now, the local Langlands correspondence for tori implies that φ_χ exists unconditionally, and has $\varphi_\chi(\Phi) = \chi \rtimes \Phi$, where on the right hand side χ is viewed as an element of $(\widehat{T}^{I_F})_{\Phi_F}$ via the Kottwitz isomorphism. But clearly $s(\pi) = \chi \rtimes \Phi$ as well.

Thus, the conjecture gives an essentially new prediction only when G is not quasi-split. In fact, its content is that the normalized transfer homomorphisms, used to define $s(\pi)$ in the general case, are really telling us what $\varphi_\pi(\Phi)$ should be. For example, if D/F is a quaternion algebra over its center F and $G = D^\times$, $J = \mathcal{O}_D^\times$, and $\pi = \mathbf{1}_{D^\times}$ (the trivial representation of $D^\times$ on $\mathbb{C}$), then Conjecture 13.1 predicts that $\varphi_\pi(\Phi) = \mathrm{diag}(q^{-1/2}, q^{1/2}) \rtimes \Phi$, where q is the cardinality of the residue field of F. This is indeed the case (cf. e.g. Lemma 13.4 or [PrRa, Thm. 4.4]).

13.2. **Proof of Conjecture 13.1 for inner forms of GL_n.** Suppose $G^* = \mathrm{GL}_n$ and that $G = \mathrm{GL}_m(D)$, where D is a central division algebra over F with $\dim_F(D) = d^2$, and m is an integer with $n = md$. We will identify $\mathrm{GL}_m(D)$ with an inner form (G, Ψ) of G^*. We will assume that LLC+ holds for the group G. Of course the local Langlands correspondence is known for GL_n, and it is also known that GL_n satisfies LLC+ (cf. [HRa, Rem. 13.1.1] or [Sch]). The local Langlands correspondence for the inner form G is also well-understood, and presumably the property LLC+ similarly holds for G. This can likely be extracted from some recent works such as [ABPS, Bad1, HiSa]. We will not verify that G satisfies LLC+ here, and instead we leave this task to another occasion.[9]

Choose $A, A^*, \psi_0 \in \Psi_M$ as in (8.1), and assume A^* is the standard diagaonal torus in $G^* = \mathrm{GL}_n$. Given $\pi \in \Pi(G/F, J)$, its supercuspidal support is $(M, \chi)_G$ for some unramified character $\chi \in X(M)$. The F-Levi subgroup $M \subset G$ (resp. $M^* := \psi_0(M) \subset G^*$) has the form

$$M \cong \prod_{i=1}^r \mathrm{GL}_{m_i}(D), \qquad (\text{resp. } M^* \cong \prod_{i=1}^r \mathrm{GL}_{n_i}, \text{ a standard Levi subgroup of } \mathrm{GL}_n)$$

for some integers m_i, n_i with $m_i d = n_i$, $\forall i$ and $\sum_i n_i = n$. It is harmless to assume that ψ_0 induces for all i an inner twisting $\mathrm{GL}_{m_i}(D) \to \mathrm{GL}_{n_i}$ which is the identity on F^s-points (only the Galois actions differ). It is also harmless to assume that $\psi_0 \in \Psi_M$ where Ψ_M is defined as in §8 using the standard upper triangular Borel subgroup $B^* \subset \mathrm{GL}_n$, and that

[9] The property LLC+ for G has recently been verified by Jon Cohen and will appear as part of his forthcoming University of Maryland PhD thesis.

$B^*_{M^*} := B^* \cap M^*$ has the form $\prod_i B^*_i$ where each B^*_i is the upper triangular Borel subgroup of GL_{n_i}.

Write $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_r$ where $\chi_i \in X(\mathrm{GL}_{m_i}(D))$. By LLC+ for G, we have equality of $\widehat{G}$-conjugacy classes

$$\varphi_\pi(\Phi) = \varphi_{\chi_1}(\Phi) \times \cdots \times \varphi_{\chi_r}(\Phi).$$

Let Nrd_i denote the reduced norm homomorphism $\mathrm{GL}_{m_i}(D) \to \mathbb{G}_m(F)$. We may write $\chi_i = \eta_i \circ \mathrm{Nrd}_i$ for a unique unramified character $\eta_i : F^\times \to \mathbb{C}^\times$. Use the same symbol η_i to denote $\eta_i(\varpi_F) \in \mathbb{C}^\times$ (here $\varpi_F \in F^\times$ is a uniformizer corresponding to Φ under the Artin reciprocity map). The Langlands dual of the homomorphism Nrd_i is the diagonal embedding $\mathrm{diag}_i : \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}_{n_i}(\mathbb{C})$. If $z_{\eta_i} \in Z^1(W_F, \mathbb{G}_m(\mathbb{C}))$ (resp. $z_{\chi_i} \in Z^1(W_F, Z(\mathrm{GL}_{n_i}(\mathbb{C})))$) is a 1-cocycle corresponding to η_i (resp. χ_i) under Langlands duality for tori (resp. quasi-characters), we have $z_{\eta_i}(\Phi) = \eta_i \in \mathbb{C}^\times$ (resp. $z_{\chi_i}(\Phi) = \mathrm{diag}_i(\eta_i) \in Z(\mathrm{GL}_{n_i}(\mathbb{C}))$).

The local Langlands correspondence for $\mathrm{GL}_{m_i}(D)$ respects twisting by unramified characters (cf. e.g. [H13, (4.0.5)]). We can view the representation χ_i as the twist of the trivial representation by the quasi-character χ_i. So in view of the above paragraph we have

$$\varphi_{\chi_i}(\Phi) = z_{\chi_i}(\Phi)\, \varphi_{\mathbf{1}_i}(\Phi) = \mathrm{diag}_i(\eta_i)\, \varphi_{\mathbf{1}_i}(\Phi),$$

where $\mathbf{1}_i$ is the trivial representation of $\mathrm{GL}_{m_i}(D)$ on $\mathbb{C}$. Thus we have

$$\varphi_\pi(\Phi) = \mathrm{diag}_1(\eta_1)\, \varphi_{\mathbf{1}_1}(\Phi) \times \cdots \times \mathrm{diag}_r(\eta_r)\, \varphi_{\mathbf{1}_r}(\Phi).$$

Lemma 13.4. *In the notation above we have $\varphi_{\mathbf{1}_i}(\Phi) = \delta_{B^*_i}^{-1/2} \rtimes \Phi$, where the modulus character is viewed as a diagonal element in $\mathrm{GL}_{n_i}(\mathbb{C})$.*

Proof. Let $\mathbf{St}_i$ (resp. $\mathbf{St}^*_i$) denote the Steinberg representation of $\mathrm{GL}_{m_i}(D)$ (resp. $\mathrm{GL}_{n_i}(F)$). Note that this has the same supercuspidal support as $\mathbf{1}_i$ (resp. $\mathbf{1}^*_i$). By LLC+ for $\mathrm{GL}_{m_i}(D)$, we see that $\varphi_{\mathbf{1}_i}(\Phi) = \varphi_{\mathbf{St}_i}(\Phi)$. The Jacquet-Langlands correspondence gives a distinguished bijection between the sets of isomorphism classes of essentially square-integrable smooth irreducible representations

$$\mathrm{JL} \,:\, \Pi^2(\mathrm{GL}_{m_i}(D)) \xrightarrow{\sim} \Pi^2(\mathrm{GL}_{n_i}(F)).$$

The Langlands parameter of $\pi_i \in \Pi^2(\mathrm{GL}_{m_i}(D))$ is that of $\mathrm{JL}(\pi_i) \in \Pi^2(\mathrm{GL}_{n_i}(F))$ (cf. e.g. [HiSa] or [Bad1]). Furthermore, $\mathrm{JL}(\mathbf{St}_i) = \mathbf{St}^*_i$ (cf. [Bad2, §7.2]). Thus we get

$$\varphi_{\mathbf{1}_i}(\Phi) = \varphi_{\mathbf{St}_i}(\Phi) = \varphi_{\mathbf{St}^*_i}(\Phi) = \delta_{B^*_i}^{-1/2} \rtimes \Phi,$$

the last equality because $\mathbf{St}^*_i$ is a quotient of $i_{B^*_i}^{\mathrm{GL}_{n_i}}(\delta_{B^*_i}^{-1/2})$. □

Therefore we have

$$\varphi_\pi(\Phi) = [\prod_i \mathrm{diag}_i(\eta_i)\, \delta_{B^*_i}^{-1/2}] \rtimes \Phi. \tag{13.2}$$

It is easy to see, using the equalities $\prod_i \delta_{B^*_i}^{-1/2} = \delta_{B^*_{M^*}}^{-1/2} = \delta_{B^*}^{-1/2} \hat{\psi}_0(\delta_P^{1/2})$ and (8.1), that (13.2) is the image of $s(\pi)$ in $[\widehat{G} \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}$. This completes the proof of Theorem 1.2. □

13.3. **Compatibility with generalized Jacquet-Langlands correspondence.** Now return to the usual notation, where G is general and is identified with an inner form (G, Ψ) of a quasi-split group G^*. Let us identify ${}^LG = {}^LG^*$ as in Remark 5.3.

Given $\pi \in \Pi(G/F, J)$, we may choose any $\pi^* \in \Pi(G^*/F, J^*)$ such that $s(\pi^*) = s(\pi)$. Note that if $J^* = K^*$, then π^* is unique, but in general it will not be.

Since $s(\pi) = s(\pi^*)$ by construction of π^*, we expect $\varphi_\pi(\Phi) = \varphi_{\pi^*}(\Phi)$. Since π and π^* are J-(resp. J^*)-spherical, φ_π and φ_{π^*} should satisfy $\varphi_\pi(I_F) = \varphi_{\pi^*}(I_F) = 1 \rtimes I_F$, and so we expect $\varphi_\pi|_{W_F} = \varphi_{\pi^*}|_{W_F}$. This is compatible with what a "generalized Jacquet-Langlands correspondence" would entail, at least on the level of infinitesimal classes (cf. [H13, §5.1]). Namely, π should give rise to the composition

$$W'_F \xrightarrow{\varphi_\pi} {}^LG =\!=\!= {}^LG^*$$

which we call φ^*, which in turn should give rise to an L-packet Π_{φ^*} for the group G^*. The map $\pi \mapsto \Pi_{\varphi^*}$ would be part of a "generalized Jacquet-Langlands correspondence". However, usually we would not expect $\pi^* \in \Pi_{\varphi^*}$. For example, if D/F and $G = D^\times$ are as above, $J = \mathcal{O}_D^\times$, $G^* = \mathrm{GL}_2$, $J^* = \mathrm{GL}_2(\mathcal{O}_F)$, and $\pi = \mathbf{1}_{D^\times}$, then $\pi^* = \mathbf{1}_{\mathrm{GL}_2(F)}$, while $\Pi_{\varphi^*} = \mathrm{JL}(\pi)$ is the Steinberg representation of $\mathrm{GL}_2(F)$.

On the other hand, if we restrict to W_F, we get an agreement of infinitesimal characters $\varphi^*|_{W_F} = \varphi_\pi|_{W_F} = \varphi_{\pi^*}|_{W_F}$. Thus, while π^* might sometimes not belong to the L-packet Π_{φ^*}, it will always belong to the infinitesimal class $\Pi_{\varphi^*|_{W_F}}$ containing Π_{φ^*}.

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