NOTES ON TATE'S p-DIVISIBLE GROUPS

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1. Statement of purpose

The aim here is simply to provide some details to some of the proofs in Tate's paper [T].

2. Tate's Section 2.2

2.1. **Lemmas about divisibility.** We say $\Gamma \to \Gamma$ is an *isogeny of the formal group* $\Gamma = \mathrm{Spf}(\mathcal{A})$ if the corresponding map $\mathcal{A} \to \mathcal{A}$ is *injective* and makes $\mathcal{A}$ free over itself of finite rank. Tate calls Γ *divisible*, if $p : \Gamma \to \Gamma$ is an isogeny. This is equivalent to $\psi : \mathcal{A} \to \mathcal{A}$ is *injective* and makes $\mathcal{A}_\psi$ free of finite rank over $\mathcal{A}$.

Lemma 2.1.1. *Suppose $\mathcal{A}_\psi$ is $\mathcal{A}$-free of rank n. Let $a_1, \dots, a_n \in \mathcal{A}$ form a basis. Then the images $\bar{a}_1, \dots, \bar{a}_n \in A_1 = \mathcal{A}/\psi(I)\mathcal{A}$ form an R-basis for A_1.*

Proof. Given $a \in \mathcal{A}$, there exist $\alpha_i = r_i + \beta_i \in R \oplus I = \mathcal{A}$ such that

$$a = \sum_i \psi(\alpha_i) a_i.$$

Reducing modulo $\psi(I)\mathcal{A}$, we get

$$\bar{a} = \sum_i \psi(r_i) \bar{a}_i$$

showing that the $\bar{a}_i$ generate A_1 over R.

If $\sum_i r_i \bar{a}_i = 0$ i.e. $\sum_i r_i a_i \in \psi(I)\mathcal{A}$, then there exist elements $\alpha_i^1 \in I$ with

$$\psi(r_1 + \alpha_1^1) a_1 + \cdots + \psi(r_n + \alpha_n^1) a_n \in \psi(I)^2 \mathcal{A}.$$

Repeating, we get elements $\alpha_i^j \in I^j$ such that

$$\psi(r_1 + \alpha_1^1 + \alpha_i^2 + \cdots) a_1 + \cdots + \psi(r_n + \alpha_n^1 + \alpha_n^2 + \cdots) a_n = 0.$$

(Using the fact that the ideals $I^j \to 0$ in the topology on $\mathcal{A}$ so that these inifinite sums converge.)

By the $\mathcal{A}$-freeness of $\mathcal{A}_\psi$, this gives

$$r_i \in R \cap \psi(I) = 0$$

for all i, proving the desired independence statement over R of the elements $\bar{a}_i \in A_1$. $\square$

Lemma 2.1.2. *Suppose $\psi : \mathcal{A} \to \mathcal{A}$ is* injective *and $\mathcal{A}_\psi$ is free over $\mathcal{A}$. Then for each ν, $\mathcal{A}_{\psi^\nu}$ is free over $\mathcal{A}$ and*

$$\mathrm{rank}_{\mathcal{A}}\mathcal{A}_{\psi^\nu} = (\mathrm{rank}_{\mathcal{A}}\mathcal{A}_\psi)^\nu.$$

Proof. If $a_1, \dots, a_n \in \mathcal{A}$ give an $\mathcal{A}$-basis for A_ψ, then the set of elements

$$\psi^{\nu-1}(a_{i_{\nu-1}}) \dots \psi(a_{i_1})a_{i_0}$$

for $\mathbf{i} = (i_{\nu-1}, \dots, i_0)$ ranging over all elements of $(\mathbb{Z}/n\mathbb{Z})^\nu$, forms an $\mathcal{A}$-basis for $\mathcal{A}_{\psi^\nu}$.

The proof is by induction on ν. The generation does not use the injectivity of ψ, but the linear-independence does. □

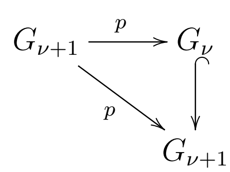
Corollary 2.1.3. *Applying Lemma 2.1.1 to both ψ and ψ^ν, and invoking Lemma 2.1.2, we get the equality*

$$\mathrm{rank}_R\mathcal{A}/\psi^\nu(I)\mathcal{A} = (\mathrm{rank}_R\mathcal{A}/\psi(I)\mathcal{A})^\nu.$$

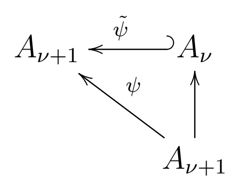
2.2. **Tate's Proposition 1.**

2.2.1. In $\Gamma \mapsto \Gamma(p)$, why is $\Gamma(p)$ p-divisible? We assume Γ is divisible. Note that by Lemma 2.1.1 and the fact that $\Gamma_{p^\nu} = \ker[p^\nu]_\Gamma$ corresponds to $\mathcal{A}/\psi^\nu(I)\mathcal{A}$, the $\mathcal{A}$-rank of $\mathcal{A}_\psi$ is the order of Γ_p, which is p^h for some h (since Γ_p is connected – cf. [Sh], p. 50). Then Corollary 2.1.3 shows that $\Gamma(p)$ is p-divisible of height h.

2.2.2. In $G \mapsto \Gamma$, why is Γ divisible? We first check that $\psi : \mathcal{A} \to \mathcal{A}$ is injective. For each ν, the diagram



corresponds to the diagram



The map $\tilde{\psi}$ is injective since $p : G_{\nu+1} \to G_\nu$ is a quotient map. This shows ψ is injective: if $\psi(a_{\nu+1})_{\nu+1} = 0$, then for all ν, we have $\tilde{\psi}(a_\nu) = 0$, which by the injectivity of $\tilde{\psi}$ implies that $a_\nu = 0$.

Next we check that $\mathcal{A}_\psi$ is free of finite rank over $\mathcal{A}$ (and the rank will be the height of G, namely $n = p^h$). Let $a_1, \dots, a_n \in \mathcal{A}$ be elements whose images in $\mathcal{A}/\psi(I)\mathcal{A}$ yield an R-basis. We claim that $a_1, \dots, a_n$ form an $\mathcal{A}$-basis for $\mathcal{A}_\psi$.

The fact that they generate is very similar to the proof of Lemma 2.1.1. Indeed, given $a \in \mathcal{A}$, for some $r_i \in R$ we have

$$a \in \sum_i r_i a_i + \psi(I)\mathcal{A}.$$

We can find some $\alpha_i^1 \in I$ for which

$$a \in \sum_i \psi(r_i + \alpha_i^1)a_i + \psi(I)^2\mathcal{A}.$$

Repeating this as in the proof of Lemma 2.1.1, we get (always with $\alpha_i^j \in I^j$)

$$a = \sum_i \psi(r_i + \alpha_i^1 + \alpha_i^2 + \cdots)a_i.$$

This shows that $\mathcal{A}_\psi$ is generated by the elements $a_1, \ldots, a_n$.

Now we prove that the a_i are independent over $\mathcal{A}$. Note first that their images clearly generate $(\mathcal{A}/\psi^\nu(I)\mathcal{A})_\psi$ over $\mathcal{A}/\psi^{\nu-1}(I)\mathcal{A}$, hence form a basis for $(\mathcal{A}/\psi^\nu(I)\mathcal{A})_\psi$ over $\mathcal{A}/\psi^{\nu-1}(I)\mathcal{A})$ as well. Why are they independent? First, the map $p : G_\nu \to G_\nu$ induces a faithfully flat *quotient* map $p : G_\nu \to G_{\nu-1}$, and thus $\psi : A_\nu \to A_\nu$ factors through the *injective* homomorphim $\tilde{\psi} : A_{\nu-1} \to A_\nu$, and A_ν is finite and flat over $A_{\nu-1}$ via $\tilde{\psi}$. In fact since $A_\nu = \mathcal{A}/\psi^\nu(I)\mathcal{A}$ is local, we see that

$$\mathcal{A}/\psi^{\nu-1}(I)\mathcal{A} \xrightarrow{\psi} \mathcal{A}/\psi^\nu(I)\mathcal{A}$$

is an injection and makes the target finite and free over the source. By comparing the R-ranks ($p^{\nu h}$ vs $p^{(\nu-1)h}$), we see that $(\mathcal{A}/\psi^\nu(I)\mathcal{A})_\psi$ has rank $n = p^h$ over $\mathcal{A}/\psi^{\nu-1}(I)\mathcal{A}$.

Now suppose we have a dependence relation

$$\psi(\alpha_1)a_1 + \cdots + \psi(\alpha_n)a_n = 0,$$

for some $\alpha_i \in \mathcal{A}$. Consider this relation modulo $\psi^\nu(I)\mathcal{A}$. The freeness result just proved shows that each $\alpha_i \in \psi^{\nu-1}(I)\mathcal{A}$. This holds for every ν. Thus each $\alpha_i = 0$, for example by Tate's Lemma 0. This completes the proof that $\psi : \Gamma \to \Gamma$ is an isogeny if Γ comes from a connected p-divisible group.

2.2.3. *Reduction of essential surjectivity in Proposition 1 to $R = k$.* In the first part of this subsection, we do not assume that the p-divisible group $G = \varinjlim_\nu \operatorname{Spec}(A_\nu)$ is connected.

Since the map of finite free R-modules $A_{\nu+1} \to A_\nu$ splits R-linearly, A is the direct product of a countable number of copies of R (as an R-module). Hence A is R-flat (as an R-module it is $R[[X]]$, which is R-flat by [AM], 10.14).

Lemma 2.2.1. *For every $n \geq 1$, we have*

$$A/\mathfrak{m}^n A = A\widehat{\otimes}_R R/\mathfrak{m}^n = \varprojlim_\nu (A_\nu \otimes_R R/\mathfrak{m}^n)$$

A similar result holds for $\mathcal{A}$ in place of A, so that

$$R[[X_1, \ldots, X_d]]\widehat{\otimes}_R R/\mathfrak{m}^n = R/\mathfrak{m}^n[[X_1, \ldots, X_d]].$$

In particular, we have $A_k = \varprojlim_\nu A_{\nu,k}$ and $\mathcal{A}_k = k[[X_1, \ldots, X_d]]$.

Proof. Let J_ν be the kernel of the projection $A \to A_\nu$. Thus $A = \varprojlim_\nu A/J_\nu$. In view of the definition of completed tensor product, our main assertion is immediate since $R/\mathfrak{m}^n$ has the discrete topology. The side assertion that $A/\mathfrak{m}^n A = A\widehat{\otimes}_R R/\mathfrak{m}^n$ is easy: use the R-flatness of A_ν and a Mittag-Leffler argument to show that

$$0 \to A\widehat{\otimes}_R \mathfrak{m}^n \to A \to A\widehat{\otimes}_R R/\mathfrak{m}^n \to 0$$

is exact; then observe that the image of $A\widehat{\otimes}_R \mathfrak{m}^n$ in A is just $\mathfrak{m}^n A$. □

Lemma 2.2.2. *Any continuous k-algebra homomorphism $\bar{\phi} : k[[X_1, \ldots, X_n]] \to A_k$ can be lifted to a continuous R-algebra homomorphism $\phi : R[[X_1, \ldots, X_n]] \to A$.*

Proof. Recall $A = \varprojlim_\nu A/J_\nu$. Let $J_{\nu,k} = J_\nu \widehat{\otimes}_R k$. Since $A \to A/J_\nu$ splits R-linearly, we have $A_k/J_{\nu,k} = A_{\nu,k}$ and thus $A_k = \varprojlim_\nu A_k/J_{\nu,k}$.

The map $\bar{\phi}$ is determined by the images of the X_i in A_k, and since A (hence A_k) is complete, the continuity/convergence amounts to saying that there exists $N = N(\nu)$, an increasing function of ν, with the property that for all i and all sufficiently large ν, we have

$$\bar{\phi}(X_i^N) \in J_{\nu,k}.$$

We may assume $N(\nu) \geq \nu$ for all ν. Let $\phi(X_i)$ be an arbitrary lift of $\bar{\phi}(X_i)$. Then these elements will determine a continuous homomorphism ϕ, provided we can prove convergence in A.

We have $\phi(X_i^N) \in J_\nu + \mathfrak{m}A$ for large ν. Thus $\phi(X_i^{N^2}) \in J_\nu + \mathfrak{m}^N A \subseteq J_\nu + \mathfrak{m}^\nu A$ for all ν large. Then the function $\nu \mapsto N(\nu)^2$ will play for ϕ the role the function $\nu \mapsto N(\nu)$ played for $\bar{\phi}$, since $J_\nu + \mathfrak{m}^\nu A \to 0$ as $\nu \mapsto \infty$, in the topology on A. □

From now on we assume $\mathrm{Spf}(A)$ is a *connected* p-divisible group. Let M_A denote the maximal ideal of A. In performing the reduction to $R = k$, we are assuming the p-divisible group $\mathrm{Spf}(A_k) = \varinjlim \mathrm{Spf}(A_{\nu,k})$ is of the form $\mathrm{Spf}(\mathcal{A}_k)$ for some d. That is, we are given a continuous isomorphism

$$k[[X_1, \ldots, X_d]] \cong A_k$$

By Lemma 2.2.2, we may lift this to a continuous homomorphism $\phi : R[[X_1, \ldots, X_d]] \to A$. By Nakayama, the composition $R[[X_1, \ldots, X_d]] \to A \to A_\nu$ is surjective for each ν. This seems to imply ϕ is surjective, but this doesn't seem to be easy to justify. Instead we take a different approach.

Write $R[[X_1, \ldots, X_d]] = \mathcal{A}$ and consider the exact sequence

$$0 \to \mathrm{Ker} \to \mathcal{A} \to A \to \mathrm{Cok} \to 0.$$

The following sequence (which is the same with the "hats" removed, hence is exact)

$$\mathcal{A}\widehat{\otimes}_R R/\mathfrak{m} \twoheadrightarrow A\widehat{\otimes}_R R/\mathfrak{m} \to \mathrm{Cok}\widehat{\otimes}_R R/\mathfrak{m} \to 0$$

shows that $\mathfrak{m}\mathrm{Cok} = \mathrm{Cok}$. But then $\mathrm{Cok} \subseteq M_A \mathrm{Cok}$. Since (A, M_A) is a local ring and Cok is finitely generated over A (by one element), we conclude that $\mathrm{Cok} = 0$.

Now the flatness of A (more precisely the flatness of every A_ν) over R implies that

$$0 \to \mathrm{Ker}\widehat{\otimes}_R R/\mathfrak{m} \to \mathcal{A}\widehat{\otimes}_R R/\mathfrak{m} \to A\widehat{\otimes}_R R/\mathfrak{m} \to 0$$

is exact. We conclude that $\mathfrak{m}\mathrm{Ker} = \mathrm{Ker}$. Since $\mathcal{A}$ is Noetherian, Ker is a finitely-generated ideal in $\mathcal{A}$. If $I = (X_1, \ldots, X_d)$, then $M = \mathfrak{m}\mathcal{A} + I$ is the maximal ideal of $\mathcal{A}$. We have $\mathrm{Ker} \subseteq \mathfrak{m}\mathrm{Ker} \subseteq M\mathrm{Ker}$, hence by Nakayama's lemma, $\mathrm{Ker} = 0$. This completes the reduction to $R = k$ that was the object of this section.

2.3. **Tate's Proposition 2.** Tate's Proposition 2 states that the discriminant ideal of A_ν over R is generated by $p^{n\nu p^{h\nu}}$, where $h = \mathrm{ht}(G)$ and $n = \dim(G)$.

2.3.1. *Discriminant identities.* Let A be an R-algebra which is free of rank n as an R-module; say $A = \oplus_{i=1}^n R\omega_i$. We define the *discriminant* to be the element of $R/(R^\times)^2$ given by

$$\delta_{A/R} = \mathrm{disc}_R(A) = \det(\mathrm{Tr}(\omega_i\omega_j)_{ij}).$$

Lemma 2.3.1. *We have the identities*

(a) $\delta_{A'\otimes_R A''/R} = (\delta_{A'/R})^{n''} \cdot (\delta_{A''/R})^{n'}$, *where* $n' = \mathrm{rk}_R(A')$ *and* $n'' = \mathrm{rk}_R(A'')$.

(b) *For* $A/A'/R$, *we have*

$$\delta_{A/R} = \delta_{A'/R}^{\mathrm{rk}_{A'}A} \cdot N_{A'/R}\delta_{A/A'}.$$

Now suppose that we have an exact sequence of finite flat commutative group schemes

$$0 \to H' \to H \to H'' \to 0.$$

with terms have orders m', m, and m'' respectively (so $m = m'm''$).

Tate uses the following lemma in his proof of his Proposition 2.

Lemma 2.3.2. $\mathrm{disc}(H) = \mathrm{disc}(H')^{m''} \cdot \mathrm{disc}(H'')^{m'}$.

Proof. (Sketch.) In order to prove the lemma, it seems necessary to extend the definition of $\delta_{A/R}$ to a broader context. We need the following ingredients, which we simply assume without proof from now on.

(i) The extension of the definition of $\delta_{A/R}$ to the context where A is faithfully flat over R and R is a product of local rings. See Conrad's Math 676 (Michigan) notes for some of this.

(ii) The Lemma 2.3.1 in this general context. I do not know a reference for this.

We have $H' \times_R H = H \times_{H''} H$ via the map $(h', h) \mapsto (h'h, h)$. We now take discriminants on both sides of the corresponding equality

$$A' \otimes_R A = A \otimes_{A''} A.$$

On the left hand side we get $(\delta_{A'/R})^m \cdot (\delta_{A/R})^{m'}$. On the right hand side we use $\delta_{A\otimes_{A''}A/A''} = (\delta_{A/A''})^{2m'}$ to get

$$\delta_{A\otimes_{A''}A/R} = (\delta_{A''/R})^{(m')^2} \cdot N_{A''/R}(\delta_{A/A''}^{2m'}).$$

Again using transitivity we have $\delta_{A/R} = \delta_{A''/R}^{m'} \cdot N_{A''/R}(\delta_{A/A''})$, so that the right hand of the last equation is

$$(\delta_{A''/R})^{-m'^2} \cdot (\delta_{A/R})^{2m'}.$$

Thus

$$(\delta_{A'/R})^m \cdot (\delta_{A/R})^{m'} = (\delta_{A''/R})^{-m'^2} \cdot (\delta_{A/R})^{2m'}$$
$$(\delta_{A'/R})^m \cdot (\delta_{A''/R})^{m'^2} = (\delta_{A/R})^{m'}.$$

Taking the m'-th roots of both side gives

$$(\delta_{A'/R})^{m''} \cdot (\delta_{A''/R})^{m'} = \delta_{A/R}.$$

□

The Lemma 2.3.2 and the obvious triviality of discriminant ideals for étale groups immediately reduced Proposition 2 to the case of connected p-divisible groups.

2.3.2. *Proof of Proposition 2 assuming Lemma 1.* We are in the situation where $G = G^\circ$ and corresponds to the formal group $\Gamma = \mathrm{Spf}(\mathcal{A})$ via the Serre-Tate correspondence. As in Proposition 1, we have $A_\nu = \mathcal{A}/J_\nu$. Consider $\mathcal{A}$ as a free module of rank $p^{h\nu}$ over itself by means of $\phi := \psi^\nu$. Consider $\mathcal{A}$ as an algebra (via ϕ) over another copy $\mathcal{A}'$ of $\mathcal{A}$. Let I' denote the augmentation ideal of $\mathcal{A}'$ (generated by the X_i'). Since $A_\nu = \mathcal{A}/I'\mathcal{A}$, it suffices to prove the discriminant ideal of $\mathcal{A}$ over $\mathcal{A}'$ is generated by the desired power of p. More precisely, suppose $a_1, \dots, a_r \in \mathcal{A}$ form a basis over $\mathcal{A}'$. Then $\bar{a}_1, \dots, \bar{a}_r \in \mathcal{A}/I'\mathcal{A}$ form an R-basis (by Lemma 2.1.1); and note $r = \mathrm{rank}_{\mathcal{A}'}\mathcal{A} = p^{h^\circ\nu}$, for $h^\circ := \mathrm{ht}(G^\circ))$. Then the reduction holds since

$$\delta_{\mathcal{A}/\mathcal{A}'} = \det(\mathrm{Tr}(a_i a_j))$$

modulo $I'\mathcal{A}$ is

$$\delta_{A_\nu/R} = \det(\mathrm{Tr}(\bar{a}_i \bar{a}_j)).$$

Next, we consider the modules of formal differentials Ω and Ω' of $\mathcal{A}$ resp. $\mathcal{A}'$. There are free modules over $\mathcal{A}$ resp. $\mathcal{A}'$ generated by the differentials of the variables dX_i resp. dX_i', $1 \leq i \leq n$. The map $\phi : \mathcal{A}' \to \mathcal{A}$ induces an $\mathcal{A}'$-linear map $d\phi : \Omega' \to \Omega$. Choosing bases in Ω' resp. Ω, we get basis element θ' resp. θ of $\Lambda^n\Omega'$ resp. $\Lambda^n\Omega$. Let $d\phi(\theta') = a\theta$, for some $a \in \mathcal{A}$.

By Lemma 1 (discussed below), we know that $\delta_{\mathcal{A}/\mathcal{A}'} = N_{\mathcal{A}/\mathcal{A}'}(a)$. Granting this, let us finish the proof of Proposition 2.

The first step is to choose a basis of translation-invariant differentials ω_i, ie. such that if $\mu : \mathcal{A} \to \mathcal{A}\widehat{\otimes}_R\mathcal{A}$ defines the formal group structure, then $d\mu : \Omega \to \Omega \oplus \Omega$ satisfies $d\mu(\omega_i) = \omega_i \oplus \omega_i$.

Let us elaborate. The identity $d\mu(\omega_i) = \omega_i \oplus \omega_i$ is a consequence of left/right invariance. For $g \in G(S)$, define *left translation* τ_g by

$$\tau_g \ : \ G \longrightarrow S \times_S G \xrightarrow{(g,\mathrm{id})} G \times_S G \xrightarrow{m} G$$

(and right translation τ'_g is similarly defined). Writing $d\mu(\omega_i) = \omega_a \oplus \omega_b$, the left-invariance $\tau_g^*\omega_i = \omega_i$ gives $\omega_i = \omega_b$. Right invariance likewise yields $\omega_i = \omega_a$. So $d\mu(\omega_i) = \omega_i \oplus \omega_i$ if ω_i is left/right invariant. On the other hand, the existence of a left (or right)-invariant basis ω_i for $\Omega_{A/R}$ is proved in [BLR], p. 100.

It follows that $d\mu^{(p)} : \Omega \to \Omega^{\oplus p}$ has $d\mu^{(p)}(\omega_i) = \omega_i^{\oplus p}$, and hence in the obvious notation we have $d\phi(\omega'_i) = p^\nu \omega_i$, whence $a = p^{\nu n}$. Proposition 2 now follows from Lemma 1 and the fact that $\mathcal{A}$ is $\mathcal{A}'$-free of rank $p^{\nu h}$.

2.3.3. *Proof of Lemma 1 assuming the existence of a certain Trace map.* We shall assume the existence and properties of Tate's trace map $\mathrm{Tr} : \Lambda^n\Omega \to \Lambda^n\Omega$:

(i) Tr is $\mathcal{A}'$-linear

(ii) $a \mapsto (\theta \mapsto \mathrm{Tr}(a\theta))$ gives an $\mathcal{A}$-linear module isomorphism

$$A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}'}(\Lambda^n\Omega, \Lambda^n\Omega')$$

(iii) If $\theta' \in \Lambda^n\Omega'$, and $x \in \mathcal{A}$, then

$$\mathrm{Tr}(x \cdot d\phi(\theta')) = (\mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}(x))\theta'.$$

Now use the basis elements $\theta \in \Lambda^n\Omega$ and $\theta' \in \Lambda^n\Omega'$ to identify $\Lambda^n\Omega \cong \mathcal{A}$ and $\Lambda^n\Omega' \cong \mathcal{A}'$. Then we can reformulate (ii) as an $\mathcal{A}$-linear isomorphism

(ii') $\mathcal{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}'}(\mathcal{A}, \mathcal{A}')$

Say $\tau \longleftrightarrow \mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}$ under the above isomorphism, which means

$$\mathrm{Tr}(\tau x\theta) = \mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}(x)\theta', \quad \forall x \in \mathcal{A}.$$

Now (iii) means $\mathrm{Tr}(xa\theta) = \mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}(x)\theta'$ for $x \in A$, and so in the presence of (ii'), (iii) can be reformulated as

(iii') $a \longleftrightarrow \mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}$ under $\mathcal{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}'}(\mathcal{A}, \mathcal{A}')$.

Therefore Lemma 1 will result from the following lemma.

Lemma 2.3.3. *If $\tau \in \mathcal{A}$ has the property of (ii'), then $\delta_{\mathcal{A}/\mathcal{A}'} = N_{\mathcal{A}/\mathcal{A}'}(\tau)$.*

Proof. Write $\mathcal{A} = \oplus_{i=1}^m \mathcal{A}' e_i$, and let $\pi_i : \mathcal{A} \to \mathcal{A}'$ be the projection onto the i-th factor. Under $\mathcal{A} \cong \mathrm{Hom}_{\mathcal{A}'}(\mathcal{A}, \mathcal{A}')$, we have $1 \longleftrightarrow f_0$ (this is the definition of f_0), and $r_i \longleftrightarrow \pi_i$ (this is the definition of r_i). It follows by $\mathcal{A}$-linearity that $r_i \longleftrightarrow r_i f_0$, so that $f_0(r_i x) = \pi_i(x)$.

Thus

$$\begin{aligned}
(N_{\mathcal{A}/\mathcal{A}'}(\tau)) &= (\det(\pi_j(\tau e_i))) \\
&= (\det(f_0(r_j \tau e_i))) \\
&= (\det(f_0(\tau r_j e_i))) \\
&= (\det(\mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}(r_j e_i))) \\
&= (\det(\mathrm{Tr}_{\mathcal{A}/\mathcal{A}'}(e_j e_i))) \\
&= \delta_{\mathcal{A}/\mathcal{A}'}.
\end{aligned}$$

Here in the penultimate equation we used the fact that $(r_j) = (a_{ij})(e_j)$ for some invertible matrix (a_{ij}) over $\mathcal{A}'$. $\square$

3. Remarks on duality for Tate's section (2.3)

3.1. **Construction of dual of a p-divisible group.** The point of characterizing G_ν as the kernel of $p^\nu : G_{\nu+\mu} \to G_{\nu+\mu}$ is that it dualizes well, allowing us to easily check that the dual of a p-divisible group is again a p-divisible group.

Indeed, we easily see that

$$\mathrm{cok}[p^\nu : G_{\nu+\mu} \to G_{\nu+\mu}] = G_{\nu+\mu}/G_\mu$$

which is isomorphic to G_ν via

$$p^\mu : G_{\nu+\mu}/G_\mu \xrightarrow{\sim} G_\nu.$$

It follows by taking duals that

$$G_\nu^\vee \xrightarrow{\sim} \ker[p^\nu : G_{\nu+\mu}^\vee \to G_{\nu+\mu}^\vee],$$

which shows that $(G_\nu^\vee, i_\nu^\vee)$ is a p-divisible group, where $i_\nu^\vee : G_\nu^\vee \to G_{\nu+1}^\vee$ is the dual of $p : G_{\nu+1} \to G_\nu$.

3.2. **Connection with dual abelian varieties.** Let X be an abelian scheme of dimension n over k, with dual abelian variety X'. Then $X'(p)$ is the Cartier dual of $X(p)$:

$$X'(p) \cong (X(p))'$$

(cf. [M]).

Let $\widetilde{X}$ be the formal completion of X along its zero section, so $\widetilde{X} = \mathrm{Spf}(k[[X_1, \dots, X_n]])$ since X is smooth of dimension n over k. Both $X(p)$ and $(X(p))'$ have height $2n$. We shall below that each has dimension n.

We have $\widetilde{X}[p^\nu] = X(p)^\circ[p^\nu]$ since $\widetilde{X}$ is supported on an infinitesimal neighborhood of the zero section. Therefore

$$\widetilde{X} \longleftrightarrow X(p)^\circ$$

under the Serre-Tate equivalence. In particular

$$\dim X(p) = \dim X(p)^\circ = \dim \widetilde{X} = n.$$

Furthermore, $X(p)^\circ$ has dimension n and height h for some integer h with $n \leq h \leq 2n$. It turns out that every such value of h is attained for some X.

4. Tate's Proposition 4

Since k is perfect the sequence

$$0 \to G_k^\circ \to G_k \to G_k^{et} \to 0$$

splits canonically. Put another way, there exists a Hopf k-algebra morphism $\bar{\phi} : A_k^\circ = k[[X_1, \dots, X_n]] \to A_k$ such that the resulting canonical morphism

$$\bar{\psi} : A_k^\circ \widehat{\otimes}_k A_k^{et} = A_k^{et}[[X_1, \dots, X_n]] \to A_k$$

is an isomorphism of Hopf k-algebras.

By Lemma 2.2.2, there is a lift ϕ for $\bar{\phi}$ (*not necessarily* a Hopf R-algebra map), and then this induces a continuous R-algebra homomorphism

$$\psi : A^\circ \widehat{\otimes}_R A^{et} \to A$$

which lifts $\bar{\psi}$.

We claim ψ is an isomorphism. This shows that as formal schemes $G = G^\circ \times G^{et}$, and hence there is a formal section of $G \to G^{et}$ (note that it is not necessarily a section of formal *groups*). This will show that for each complete ring S we can plug into these functors, the sequence

$$0 \to G^\circ(S) \to G(S) \to G^{et}(S) \to 0$$

is exact.

Let $K := \mathrm{Ker}(\psi)$ and $C := \mathrm{Cok}(\psi)$. If $C \neq 0$, then choose a maximal ideal $\mathfrak{M}$ of A such that $C_{\mathfrak{M}} \neq 0$. The surjectivity of $\bar{\psi}$ implies that $C/\mathfrak{m}C = 0$, i.e. $C = \mathfrak{m}C$. Then we also have $C_{\mathfrak{M}} = \mathfrak{m}C_{\mathfrak{M}}$. Now $C_{\mathfrak{M}}$ is finitely generated (by one element) over the local ring $A_{\mathfrak{M}}$, and $C_{\mathfrak{M}} = \mathfrak{m}C_{\mathfrak{M}} \subseteq \mathfrak{M}C_{\mathfrak{M}}$. Thus by Nakayama, $C_{\mathfrak{M}} = 0$, a contradiction. Thus $C = 0$.

Now we prove that $K = 0$. Set $\widehat{A} := A^\circ \widehat{\otimes}_R A^{et}$. As above, since A is R-flat, we get an exact sequence

$$0 \to K_k \to \widehat{A}_k \to A_k \to 0$$

hence $K = \mathfrak{m}K$. Now we'd like to argue as for C to show that $K = 0$; however, since $\widehat{A}$ need not be Noetherian, we can't say K is a finitely generated ideal in $\widehat{A}$, so we need a different argument. Denote by $\widehat{J}_\nu \subset \widehat{A}$ the obvious family of ideals such that $\widehat{A} = \varprojlim_\nu \widehat{A}/\widehat{J}_\nu$. Note that if $\widetilde{J}_\nu$ denotes the image of $\widehat{J}_\nu$ in A, then we have an exact sequence

$$0 \to K/K \cap \widehat{J}_\nu \to \widehat{A}/\widehat{J}_\nu \to A/\widetilde{J}_\nu \to 0.$$

Now $K/K \cap \widehat{J}_\nu$ is a finitely-generated ideal in the Noetherian ring $\widehat{A}/\widehat{J}_\nu$, hence the equality $\mathfrak{m}(K/K \cap \widehat{J}_\nu) = K/K \cap \widehat{J}_\nu$ implies by the argument for C above that $K = K \cap \widehat{J}_\nu$. (This works even though $\widehat{A}$ like A is not local; we need to localize at its maximal ideals.) Then we see that $K \subseteq \bigcap_\nu \widehat{J}_\nu = 0$, and $K = 0$ as desired.

5. Tate's Corollary 1 to Proposition 4

An easy diagram chase reduces us to the separate cases where G is étale or connected. First suppose G is étale. Then as noted earlier in Tate's section 2.4, we have

$$G(S) = \varinjlim_\nu G_\nu(S/\mathfrak{m}S).$$

It is enough to show that for $x \in G_\nu(S/\mathfrak{m}S)$, there is a finite extension $S' \supset S$ of the same form as S and an element $y \in G_{\nu+1}(S'/\mathfrak{m}S')$ such that $py = x$. Let k_S denote the residue field of S (equivalently, of $S/\mathfrak{m}S$). By the infinitesimal lifting criterion for étale schemes, we have $G_\nu(S/\mathfrak{m}S) = G_\nu(k_S)$. By the surjectivity of $G_{\nu+1} \to G_\nu$, there is some finite extension $k' \supset k_S$ and an element $y \in G_{\nu+1}(k')$ with $py = x$ in $G_\nu(k')$. Now let S' be a complete valuation ring containing S and having residue field k'. By the infinitesimal lifting property, we have $y \in G_{\nu+1}(S'/\mathfrak{m}S') = G_{\nu+1}(k')$, and $py = x$ holds in $G_\nu(S'/\mathfrak{m}S')$, as desired.

Now we suppose G is connected. Let Γ be the associated divisible commutative formal group, with ring $\mathcal{A} = R[[X_1, \dots, X_n]]$. We identify $G(S)$ with the set $\mathrm{Hom}_{\mathrm{conts}}(\mathcal{A}, S)$ of continuous R-algebra homomorphisms $\phi : \mathcal{A} \to S$. We need to find a finite extension of complete valuation rings $i : S \hookrightarrow S'$ and a homomorphism $\phi' : \mathcal{A} \to S'$ for which $\phi' \circ \psi = i \circ \phi$. Since $\psi : \mathcal{A} \hookrightarrow \mathcal{A}$ makes $\mathcal{A}$ finite and free over itself, we may take for the ϕ' the canonical map $\mathcal{A} \to \mathcal{A}\widehat{\otimes}_{\psi,\mathcal{A}} S$ in the push-out diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi'} & \mathcal{A}\widehat{\otimes}_{\psi,\mathcal{A}} S \\ {\scriptstyle\psi}\uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{\phi} & S \end{array}$$

6. Tate's Main Theorem (4.2)

We just make a few remarks about a few points in the proof.

Use of disciminants in Proof of Cor. 2. Tate uses the fact that if $B_\nu \subseteq A_\nu$ and $\mathrm{disc}_R(B_\nu) = \mathrm{disc}_R(A_\nu)$, then $B_\nu = A_\nu$. This follows by choosing a "stacked basis" $\omega_1, \dots, \omega_n$ resp. $\pi^{r_1}\omega_1, \dots, \pi^{r_n}\omega_n$ for A_ν resp. B_ν, where each $r_i \geq 0$. Since

$$\det(\mathrm{Tr}(\pi^{r_i}\omega_i\, \pi^{r_j}\omega_j)) = c^2 \det(\mathrm{Tr}(\omega_i\omega_j)),$$

where $c = \pi^{\sum_i r_i}$, we see that $\mathrm{disc}_R(B_\nu) = \mathrm{disc}_R(A_\nu)$ implies that $r_i = 0$ for all i, hence $B_\nu = A_\nu$.

Why is M assumed to be a $\mathbb{Z}_p$-summand in Prop. 12? Given $M \subset T(F)$, we need to find a corresponding p-divisible group $E_* \subset F \otimes_R K$. Because M is a direct summand, the Galois module $M/p^\nu M$ is contained in $T(F)/p^\nu T(F) = F_\nu(\bar{K})$. We define the étale K-group $E_{*\nu}$ by the equality of Galois modules $E_{*\nu}(\bar{K}) := M/p^\nu M$.

Why the algebras D_i are stationary Recall that R is a PID in this discussion.

Tate defines $E_\nu = \mathrm{Spec}(A_\nu)$ where $A_\nu := u_\nu(B_\nu) \subset A_{*\nu}$. It follows that each A_ν is a finite-rank and free Hopf algebra over R. Since

$$E_{i+1}/E_i = \mathrm{Spec}(D_i)$$

by definition of quotient we have $D_i \subset A_{i+1}$ hence D_i is also finite-free over R.

The maps induced by p

$$E_{i+\nu+1}/E_{i+1} \to E_{i+\nu}/E_i$$

(generically isomorphisms) induce maps

$$D_i \to D_{i+1}$$

which are all isomorphisms upon tensoring by K, hence are all injective and the D_i can be viewed as an increasing sequence of R-orders in a common separable K-algebra $D_i \otimes_R K$. Thus the D_i are all contained in the integral closure of R in $D_i \otimes_R K$.

Lemma 6.0.1. *If R is a normal Noetherian domain, and $\tilde{R}$ is its integral closure in a finite-dimensional separable K-algebra, then $\tilde{R}$ is a Noetherian R-module.*

Proof. Adapt 5.17 of [AM], which handles the case of separable *field* extensions of K. □

Since the integral closure of R in $D_i \otimes_R K$ is Noetherian R-module, it follows that there exists an i_0 such that $D_i = D_{i+1}$ for all $i \geq i_0$.

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