UCLA

Colloquium

Saturation problems via affine Grassmannians and triangles in buildings

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On www.arxiv.org:

[KLM] M. Kapovich, B. Leeb, J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, math.RT/0210256.

[H] T. Haines, *Structure constants for Hecke and representation rings*, math.RT/0304176.

Related Preprint (not yet on server):

M. Kapovich, B. Leeb, J. Millson, *Polygons in symmetric spaces and buildings*.

Tensor products of representations

G - complex linear algebraic group, T a maximal torus. $X^*(T) =$ weights.

Theory of highest weight: Any dominant weight $\lambda \mapsto V_\lambda$ an irreducible representation of G, with "highest weight" λ.

Example: $\mathrm{GL}_n(\mathbb{C}) \supset$ diag. torus; dominant weights $\leftrightarrow$ non-increasing n-tuples $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$.

$(1^r, 0^{n-r}) \leftrightarrow \Lambda^r(\mathbb{C}^n)$, and $(k, 0^{n-1}) \leftrightarrow \mathrm{Sym}^k(\mathbb{C}^n)$.

Question: How to describe

$$\begin{aligned} R^3_G &:= \{(\alpha, \beta, \gamma) \in X^*(T)^3_{\mathrm{dom}} \mid (V_\alpha \otimes V_\beta \otimes V_\gamma)^G \neq 0\} \\ &= \{\cdots \mid \text{triv. rep. } \mathbb{I} \in V_\alpha \otimes V_\beta \otimes V_\gamma\} \\ &= \{\cdots \mid V_{\gamma^*} \in V_\alpha \otimes V_\beta\}. \end{aligned}$$

Here γ^* is the dominant weight indexing the (irred.) dual of V_γ.

$\exists$ Many Methods to answer question (or more precisely, compute the multiplicity $n^{\lambda}_{\alpha,\beta}$, the number of times V_γ occurs in $V_\alpha \otimes V_\beta$):

– (GL_n) Littlewood-Richardson rule

– Berenstein-Zelevinsky polytopes ($\rightarrow$ honycomb model for GL_n (Knutson-Tao))

– Littelmann path model

– Kashiwara's crystal bases/graphs

Also: New algorithm due to [KLM], in terms of multiplication in Hecke rings

$R^3_{\mathsf{GL}_n}$ is the set of *integral points* in the following (identical) polyhedral cones in $(\mathbb{R}^n)^3$:

Eigenvalues of a sum The set of triples (α, β, γ) of dominant vectors in $\mathbb{R}^n$ (entries in non-increasing order) such that there exist Hermitian matrices A, B, C such that the set of eigenvalues of A resp. B resp. C is α, resp. β, resp. γ, and

$$A + B + C = 0.$$

Generalized triangle inequalities The triples as above which satisfy a system of inequalities defined using Schubert calculus:

Let $\Delta \subset \mathbb{R}^n \cong X^*(T) \otimes \mathbb{R}$ denote the cone of dominant (real) weights. Fundamental coweight $(1^i, 0^{n-i}) = \lambda \in X_*(T) =$ a linear functional on Δ.

$\lambda \mapsto$ parabolic $P_\lambda \mapsto$ Schubert variety GL_n/P_λ.

GTI's: for each λ and each triple $w_1, w_2, w_3 \in W\lambda \subset X_*(T)$, impose the inequality

$$w_1(\alpha) + w_2(\beta) + w_3(\gamma) \leq 0$$

whenever

$$[X_{w_1}] \cdot [X_{w_2}] \cdot [X_{w_3}] = [\text{pt}] \ \text{ in } H_*(G/P).$$

GTI's for general G defined similarly. In that context get

Theorem 1 (Leeb-Millson) (α, β, γ) *satisfy GTI's if and only if there is a triangle in the symmetric space* $\mathcal{G}(\mathbb{C})/K$ *having "geodesic side lengths"* α, β, γ. *(Here* $\mathcal{G}$ *is such that* $G = \widehat{\mathcal{G}}$*, and* $K =$ *a maximal cpt. subgroup.)*

Rank 1 symm. space = disc, $\Delta = \mathbb{R}_{>0}$ = usual notion of "side length". [Picture].

In general, by Cartan decomposition $\mathcal{G}(\mathbb{C}) = K\exp(\Delta)K$: directed geodesic $\overline{x_1 x_2} \in K \cdot \overline{0\exp(\delta)}$, for unique $\delta \in \Delta$.

General features of R_G^3

- R_G^3 is a *semi-group*: $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in R_G^3 \Rightarrow (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2) \in R_G^3$.

- (Knutson-Tao) If $G = \mathsf{GL}_n$, then R_G^3 is *saturated*: For $N > 1$,
$$(N\alpha, N\beta, N\gamma) \in R_G^3 \Rightarrow (\alpha, \beta, \gamma) \in R_G^3.$$

(Used in proof of above description of $R^3_{\mathsf{GL}_n}$ in terms of GTI's.)

For general G, R_G^3 is usually NOT saturated.

Questions: When exactly does the saturation property hold for other groups? Is there a "uniform" geometric approach to this?

Saturation Problems for Hecke algebras

New notation: G - linear alg. group over finite field $\mathbb{F}_q$, T max. torus. Old G, T are now the Langlands dual (over $\mathbb{C}$) $\widehat{G}, \widehat{T}$ of the new G, T. $X^*(T) = X_*(\widehat{T})$, $X_*(T) = X^*(\widehat{T})$.

$G(\mathbb{F}_q((t)))$:= "loop group LG of G" $\supset$ max'l compact $K := G(\mathbb{F}_q[[t]])$. Then have Hecke algebra $H = C_c(K \backslash G(\mathbb{F}_q((t)))/K)$ with convolution $*$.

$\alpha \in X_*(T)_{\text{dom}}$ provides a *basis element* for H:

$$f_\alpha := \text{char}(K\alpha(t)K).$$

[By Cartan decomposition: $K \backslash LG/K \leftrightarrow X_*(T)_{\text{dom}}$. ($\text{GL}_n$:"elem. divisors").]

Definition $f_\alpha * f_\beta * f_\gamma = \sum_\lambda c^\lambda_{\alpha,\beta,\gamma} f_\lambda$.

Question: Describe

$$H^3_G := \{(\alpha, \beta, \gamma) \in X_*(T)^3_{\text{dom}} \mid c^0_{\alpha,\beta,\gamma} \neq 0\}.$$

Hecke saturation

Theorem 2 (KLM) *For each group G, $\exists\, k = k(G) \in \mathbb{N}$ such that for $N > 1$, $(N\alpha, N\beta, N\gamma) \in H^3_G \Rightarrow (k\alpha, k\beta, k\gamma) \in H^3_G$.*

Corollary 1 *For $H^3_{\mathsf{GL}_n}$ is saturated ($k(\mathsf{GL}_n) = 1$).*

Idea of proof: The GTI's also characterize side lengths of triangles in Bruhat-Tits building for LG. The GTI's are *homogeneous*, hence solution set in Δ^3 saturated (in strong sense). But want triangles in building with special points as vertices. This is where $k(G)$ comes in. [Picture]

How the sets are related

New notation: R^3_G written instead of

$R^3_{\widehat{G}} = \{(\alpha, \beta, \gamma) \in (X^*(\widehat{T})_{\mathsf{dom}})^3 \mid (V_\alpha \times V_\beta \otimes V_\gamma)^{\widehat{G}} \neq 0\}$.

Combinatorial methods give:

Theorem 3 (KLM) *For general G, $R^3_G \subset H^3_G$.*

Theorem 4 (P. Hall) $H^3_{\mathsf{GL}_n} \subset R^3_{\mathsf{GL}_n}$.

Corollary 2 $R^3_{\mathsf{GL}_n}$ *is saturated (!).*

[KLM] result uses Lusztig's q-analogue of weight-multiplicity formula (tricky, but valid for every group.)

[P. Hall] result uses combinatorics of Hall algebras – special to GL_n.

In general H^3_G is strictly bigger than R^3_G (e.g. G_2, SO(5)).

Affine Grassmannians

Let's define the *Affine Grassmannian* (for GL_n). Let $k = \bar{\mathbb{F}}_q$, $F = k((t))$, $\mathcal{O} = k[[t]]$.

Define

$\mathcal{Q}(k) = \{\mathcal{O}-\text{lattices in } F^n\} = G(k((t)))/G(k[[t]])$,

an ind-scheme defined over $\mathbb{F}_q$.

Finite-dimensional pieces: inv gives "distance" between two lattices:

$\mathrm{inv}(L, L') := \lambda \in X_*(T)_{\mathrm{dom}}$, if $L = g\mathcal{O}^n$, $L' = g'\mathcal{O}^n$, and $g^{-1}g' \in K(\mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n}))K$.

Affine Schubert variety: $\bar{\mathcal{Q}}_\mu = \{L \in \mathcal{Q} \mid \mathrm{inv}(\mathcal{O}^n, L) \leq \mu\}$. ($\leq$ the standard partial order on dominant coweights...) These are fin.-dim'l projective varieties, over $\mathbb{F}_q$.

- K-orbits are the Schubert cells $\mathcal{Q}_\mu$

- $\mathcal{Q} = \coprod_\mu \mathcal{Q}_\mu$, and the boundary of $\bar{\mathcal{Q}}_\mu$ is the union of the $\mathcal{Q}_\lambda$'s, where $\lambda < \mu$.

Can do all this for general G (G-bundles on curves, with trivialization outside a fixed point).

Consider $P_K(\mathcal{Q})$, the K-*equivariant perverse sheaves* on $\mathcal{Q}$. (Geometric analogue of Hecke ring).

$P_K(\mathcal{Q})$ is a semi-simple, abelian category. Simple objects: the *intersection complexes* $\mathrm{IC}(\bar{\mathcal{Q}}_\mu)$.

(IC complexes compute intersection cohomology...)

There is a convolution $* : P_K(\mathcal{Q}) \times P_K(\mathcal{Q}) \to P_K(\mathcal{Q})$, making $P_K(\mathcal{Q})$ a *Tannakian* category.

By Drinfeld, Ginzburg, Mirkovic-Vilonen, Ngô-Polo... , we have the famous geometric version of the usual Satake isomorphism from the Hecke algebra of LG to the rep. ring of $\widehat{G}$.

Theorem 5 (Geometric Satake Isomorphism) *There is an isomorphism* $(\mathrm{Rep}(\hat{\mathrm{G}}), \otimes) \cong (\mathrm{P_K}(\mathcal{Q}), *)$, *such that* V_μ *corresponds to* $\mathrm{IC}(\bar{\mathcal{Q}}_\mu)$.

Corollary 3 *in* $P_K(\mathcal{Q})$,

$$\mathrm{IC}_{\mu_1} * \cdots * \mathrm{IC}_{\mu_r} = \sum_\lambda \dim(V^\lambda_{\mu_\bullet})\ \mathrm{IC}_\lambda.$$

Geometric reformulation

$\lambda \in X_*(T)_{\text{dom}}$, $\mu_\bullet = (\mu_1, \ldots, \mu_r) \in X_*(T)^r_{\text{dom}}$. Write $|\mu_\bullet| := \sum_i \mu_i$.

$\mathbf{Rep}(\mu_\bullet, \lambda)$: $V^\lambda_{\mu_\bullet} \neq 0$, where $V_{\mu_1} \otimes \cdots \otimes V_{\mu_r} = \oplus_{\lambda \leq |\mu_\bullet|} V^\lambda_{\mu_\bullet} \otimes V_\lambda$.

$\mathbf{Hecke}(\mu_\bullet, \lambda)$: $c^\lambda_{\mu_\bullet} \neq 0$, where $f_{\mu_1} * \cdots * f_{\mu_r} = \sum_{\lambda \leq |\mu_\bullet|} c^\lambda_{\mu_\bullet} f_\lambda$, where $f_\mu = \text{char}(K t_\mu K)$.

Define the twisted product $\tilde{\mathcal{Q}}_{\mu_\bullet}$ to be

$\{L_\bullet = (L_1, \ldots, L_r) \in \mathcal{Q}^r \mid \text{inv}(L_{i-1}, L_i) \leq \mu_i, \forall i\}$.

Definition

$$m_{\mu_\bullet} : \tilde{\mathcal{Q}}_{\mu_\bullet} \to \bar{\mathcal{Q}}_{|\mu_\bullet|}$$

given by $L_\bullet \mapsto L_r$.

This is the geometric analogue of convolution in Hecke algebra. (Used to define convolution in category $P_K(\mathcal{Q})$ due to Drinfeld, V. Ginzburg, and studied in geometric Langlands program...)

Key fact used in this definition:

Theorem 6 (Mirkovic-Vilonen, Ngô-Polo) *The birational morphism $m_{\mu_\bullet}$ is locally trivial and* semi-small, *in the sense of Goresky-MacPherson.*

The semi-smallness means that the fibers are not too large: if $y \in \mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$, then

$$\dim(m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu'}) \leq \frac{1}{2}(\dim(\mathcal{Q}_{\mu'_\bullet}) - \dim(\mathcal{Q}_\lambda));$$

where RHS is also $\langle \rho, |\mu'_\bullet| - \lambda \rangle$, where $\rho =$ half-sum of positive roots.

Can now reformulate $\mathbf{Rep}(\mu_\bullet, \lambda)$ and $\mathbf{Hecke}(\mu_\bullet, \lambda)$ in terms of *fibers of* $m_{\mu_\bullet}$:

Theorem 7 (H) *1)* $V^\lambda_{\mu_\bullet} \neq 0$ *iff* $\dim(m^{-1}_{\mu_\bullet}(y)) = \langle \rho, |\mu_\bullet| - \lambda \rangle$. *Any irreducible component of such max'l dimension meets the open stratum* $\mathcal{Q}_{\mu_\bullet}$.

2) If y *is* $\mathbb{F}_q$*-rational, then* $c^\lambda_{\mu_\bullet} \neq 0$ *iff* $m^{-1}_{\mu_\bullet}(y) \cap \mathcal{Q}_{\mu_\bullet} \neq \emptyset$.

Get: geometric proof of Theorem 3. To prove above result, use

$$Rm_{\mu_\bullet,*}(\mathrm{IC}(\tilde{\mathcal{Q}}_{\mu_\bullet})) = \mathrm{IC}_{\mu_1} * \cdots * \mathrm{IC}_{\mu_r}$$

and basic properties of IC, to show that $V^\lambda_{\mu_\bullet}$ *has a basis in canonical correspondence with the irreducible components of max'l possible dimension in the fiber* $m^{-1}_{\mu_\bullet}(y)$.

Also get

- proof of Theorem 4: need

  Lemma 1 (H) *For* GL_n*, if* μ_i *all* minuscule, *then all fibers of* $m_{\mu_\bullet}$ *are* equidimensional*. [use Spaltenstein-Springer varieties]*

- For general G, Weil conjectures give [KLM] formula:

  $c^{\lambda}_{\mu_\bullet}(q) = \dim(V^{\lambda}_{\mu_\bullet}) q^{\langle \rho, |\mu_\bullet| - \lambda \rangle} + \{\text{lower deg terms}\}.$

  This gives a new (not very fast) algorithm to compute $\dim(V^{\lambda}_{\mu_\bullet})$!

Generalizations?

Above methods work well for arbitrary groups provided λ, μ_i are *sums of minuscule coweights*.

I have "almost proved" the following natural

Conjecture 1 *Suppose* λ, μ_i *are sums of minuscule coweights. Then for* $N > 1$,

$$\dim(V^{N\lambda}_{N\mu_\bullet}) \neq 0 \Rightarrow \dim(V^{\lambda}_{\mu_\bullet}) \neq 0.$$

Groups with minuscule coweights: PGL(n+1) (n); SO($2n+1$) (1); GSp($2n$) (1); SO($2n$) (3); E_6 (2); E_7 (1).

THE END.