UCLA

Number theory seminar

Local zeta functions for some Shimura varieties with tame bad reduction

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Local zeta functions

Consider $E/\mathbb{Q}_p$, and a smooth d-dimensional variety X/E, with model over $\mathcal{O}_E$. Let $\mathfrak{p} =$ prime ideal of $\mathcal{O}_E$.

Definition We define $Z_{\mathfrak{p}}(s, X)$ to be

$$\prod_{i=0}^{2d} \mathrm{det}(1 - N\mathfrak{p}^{-s}\Phi_{\mathfrak{p}}; H_c^i(X \times_E \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell)^{\Gamma_{\mathfrak{p}}^0})^{(-1)^{i+1}},$$

where

- $\ell \neq p$,
- $\Phi_{\mathfrak{p}} =$ geom. Frobenius,
- inertia subgroup $= \Gamma_{\mathfrak{p}}^0 \subset \Gamma_{\mathfrak{p}} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$

Good reduction implies: can forget $(\cdot)^{\Gamma^0}$ and pass to counting rational points over fields $\mathbb{F}_q$...

For varieties defined over number fields, take product of above over all finite places (and take something at infinite places...). We want to understand *analytic properties*, e.g. analytic continuation, functional equation, and special values. For latter two, must consider finite number of places with *bad reduction*.

Definition above is likely "correct": e.g., $\exists$ heuristic argument indicating $(\cdot)^{\Gamma^0}$ ensures functional equation...(later).

Basic problem in Langlands program: for Shimura varieties, express $Z_{\mathfrak{p}}(s, X)$ in terms of local factors of *automorphic L-functions*.

Local L-functions

$\pi_p =$ irred. admissible rep. of $G(\mathbb{Q}_p)$, having local Langlands parameter $\phi_{\pi_p} : W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G := W_{\mathbb{Q}_p} \ltimes \widehat{G}$. Let $r = (r, V)$ be an algebraic rep. of ${}^L G$.

Definition We define $L_p(s, \pi_p, r)$ to be

$$\det(1 - p^{-s} r\phi_{\pi_p}(\Phi \times \begin{bmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{bmatrix}); (\ker N)^{\Gamma^0})^{-1},$$

- $\Phi \in W_{\mathbb{Q}_p}$ a geom. Frobenius,
- $N := r\phi_{\pi_p}(1 \times \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$, nilpotent operator on V,
- Γ^0-action on $\ker(N) \subset V$ is via $r\phi_{\pi_p}$ restricted to $\Gamma^0 \times \mathrm{id} \subset W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$.

Note π_p *spherical* implies: $(\mathrm{ker} N)^{\Gamma^0} = V$, and can express in terms of Satake parameter of π_p:

$$\det(1 - p^{-s} r(\mathrm{Sat.}(\pi_p)); V)^{-1},$$

("automorphic analogue of good reduction").

Tame bad reduction

Take X over number field $\mathbb{E}$. We say X has *tame bad reduction at* $\mathfrak{p}$ provided that $\Gamma^0_{\mathfrak{p}}$ acts *unipotently* on cohomology of $X \times_{\mathbb{E}_{\mathfrak{p}}} \bar{\mathbb{Q}}_p$ (action by functoriality). This will happen for Shimura varieties with *Iwahori* (more generally *parahoric*) level structure at p (later...).

The automorphic analogue is: $\pi_f = \otimes_v \pi_v$ satisfies $\phi_{\pi_p}(\Gamma^0_p) = \mathrm{id}$. (Thus, for any (r, V) we have $(\mathrm{ker} N)^{\Gamma^0} = \mathrm{ker} N$.) Conjecturally, this happens at least when π_p has *Iwahori fixed vectors*.

Shimura varieties

$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash[G(\mathbb{A}_f) \times X_\infty]/K$, a variety defined over reflex field $\mathbb{E}$. $S_K = \mathrm{Sh}(G, h, K)$, where $h : \mathbb{C}^\times \to G_\mathbb{R}$ and $X_\infty =$ the $G(\mathbb{R})$-conj. class of h.

$h \mapsto \mu$, a *minuscule* cocharacter of $G_{\bar{E}}$.

Iwahori level structure at p: $K = K^p K_p$, $K_p \subset G(\mathbb{Q}_p)$ an Iwahori subgroup.

Consider PEL Shimura varieties S_K (moduli spaces of abelian varieties with add. structure...). For $\mathfrak{p} \in \mathbb{E}$ dividing p, and $E := \mathbb{E}_\mathfrak{p}$, get a model over $\mathcal{O}_E$ by posing suitable moduli problem over $\mathcal{O}_E$. We can do this in case of Iwahori level structure at p (but not deeper level structure).

Prototype: $Y_0(p)$, attached to Iwahori $\begin{pmatrix} * & * \\ p* & * \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}_p)$. Moduli space: $(E_1 \to E_2)$, degree p isogenies of elliptic curves. Special fiber is union of two smooth curves, intersecting transversally at the supersingular points.

In general, the moduli space parametrizes chains $(A_1, \lambda_1, \iota_1, \eta) \to (A_2, \lambda_2, \iota_2, \eta) \to \cdots$ of p-isogenies between polarized abelian varieties with additional structure (still makes sense over $\mathcal{O}_E$; singularities in special fiber now much more complicated...)

Goal: understand something about $Z(s, S_K)$ in case of Iwahori level structure.

First, we summarize Langlands' strategy in case $Y(N)$, where $p \nmid N$ (good reduction). (Adding cusps gives $X(N)$...)

Langlands' strategy, for $Y(N)$, $p \nmid N$

Here $G = \mathrm{GL}_2$, and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.

1) Write $\mathrm{Tr}(\Phi^j; H_c^\bullet(S_K \times_E \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell))$ in form

$$\sum_{\gamma_0} \sum_{\gamma,\delta} (\mathrm{vol}) \mathrm{O}_\gamma(f^p) \mathrm{TO}_{\delta\sigma}(\phi_j)$$

2) Fundamental lemma: $\mathrm{TO}_{\delta\sigma}(\phi_j) = \mathrm{O}_{N\delta}(b\phi_j)$
3) use Arthur-Selberg trace formula

$$\sum_{\gamma_0} (\mathrm{vol}) \mathrm{O}_{\gamma_0}(f) + \cdots = \sum_{\pi} m(\pi) \mathrm{Tr}\, \pi(f) + \cdots .$$

Explanations: 1) $(\gamma_0, \gamma, \delta) \in G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^j})$, such that γ (resp. $N\delta$) is locally stably conjugate to γ_0. $\sigma =$ can. gen. of $\mathrm{Gal}(\mathbb{Q}_{p^j}/\mathbb{Q}_p)$, and

$$\mathrm{TO}_{\delta\sigma}(\phi) = \int_{G_{\delta\sigma} \backslash G(\mathbb{Q}_{p^j})} \phi(x^{-1}\delta\sigma(x)) \frac{dx}{dx_{\delta\sigma}}.$$

Use Lefschetz trace formula, and count points over $\mathbb{F}_q$ using Honda-Tate theory: s.s. conjugacy classes $\gamma_0 \leftrightarrow$ isogeny classes of elliptic curves. Then count points (E, η) where E ranges over a given isogeny class (η is a full level-N structure on E).

$$\phi_j = \text{char}(\text{GL}_2(\mathbb{Z}_{p^j}) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{GL}_2(\mathbb{Z}_{p^j})) \in H(\text{GL}_2(\mathbb{Q}_{p^j})),$$

2) $b : H(\text{GL}_2(\mathbb{Q}_{p^j})) \to H(\text{GL}_2(\mathbb{Q}_p))$ is "base change" homomorphism for spherical Hecke algebras.

Even with good reduction, get complications in higher dimensions from non-compactness and endoscopy. However, these problems don't arise for Kottwitz simple Shimura varieties (defined later...). Still, even for these varieties, new complications emerge when there is tame bad reduction.

Problems in bad reduction case

(A) non-trival inertia action on $H^i(S_K \times_E \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell)$.
(B) Assume $S_K/\mathcal{O}_E$ *proper*. Then $H^i(S_K \times_E \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell) = H^i(S_K \times_{\mathcal{O}_E} \bar{\mathbb{F}}_p, \mathsf{R}\Psi(\bar{\mathbb{Q}}_\ell))$, where $\mathsf{R}\Psi(\bar{\mathbb{Q}}_\ell) \in D(S_K \times \bar{\mathbb{F}}_p)$ is the *sheaf of nearby cycles*, a complex of sheaves whose complexity measures the singularities in the special fiber.

To (temporarily) circumvent (A), we work with *semi-simple* trace: If V is an ℓ-adic rep. of $\Gamma := \mathsf{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\exists$ finite Γ-stable filtration $\cdots V_{k-1} \subset V_k \subset \cdots \subset V$ such that inertia Γ^0 acts through finite quotient on $\oplus_k gr_k V_\bullet$. Then set

$$\mathsf{Tr}^{ss}(\Phi, V) := \sum_k \mathsf{Tr}(\Phi, (gr_k V_\bullet)^{\Gamma^0}).$$

Semi-simple trace extends to give function-sheaf correspondence à la Grothendieck. In particular, there is a Leftschetz trace formula for it.

We can define $Z^{ss}(s, X)$ and $L^{ss}(s, \pi_p, r)$ using Tr^{ss} in place of Tr. To express Z^{ss} in terms of various $L^{ss}(s, \pi_p, r)$, for *simple Shimura varieties*, we can imitate Langlands.

1) Via LTF, write $\mathrm{Tr}^{ss}(\Phi^j, H^\bullet(S_K \times \bar{\mathbb{F}}_p, \mathrm{R}\Psi))$ as

$$\sum_{x \in \mathrm{Fix}(\Phi^j, S_K(\bar{\mathbb{F}}_p))} \mathrm{Tr}^{ss}(\Phi^j, \mathrm{R}\Psi_x).$$

Via Honda-Tate theory, write latter in form

$$\sum_{\gamma_0} \sum_{\gamma,\delta} (\pm \mathrm{vol}) \mathrm{O}_\gamma(f^p) \mathrm{TO}_{\delta\sigma}(\phi_j),$$

where ϕ_j is a suitable function in $H(G(\mathbb{Q}_{p^j})//I_j)$.

2) Fundamental lemma: $\mathrm{STO}_{\delta\sigma}(\phi_j) = \mathrm{SO}_{N\delta}(b\phi_j)$, where $b\phi_j$ is “base-change” of ϕ_j,

3) Use Arthur-Selberg trace formula (simple since G/A_G anisotropic...)

The main difficulties are 1) and 2) (the stabilization leading to 3) being just like Kottwitz’s work in good reduction case).

Theorem 1 (H., B.C. Ngô) *Let $S_K = \mathrm{Sh}(G, \mu, K)$ be a simple Shimura variety with Iwahori (more generally, parahoric) level structure at p. Let $r_\mu : {}^LG \to \mathrm{Aut}(V_\mu)$ be the irreducible representation of ${}^LG$ with highest weight μ. Let $d = \dim(S_K)$. Then*

$$Z^{ss}_{\mathfrak{p}}(s, S_K) = \prod_{\pi_f} L^{ss}_p(s - \frac{d}{2}, \pi_f, r_\mu)^{a(\pi_f)\dim(\pi_f^K)},$$

where π_f ranges over irred. adm. reps. of $G(\mathbb{A}_f)$, the integer number $a(\pi_f)$ is given by

$$a(\pi_f) = \sum_{\pi_\infty \in \Pi_\infty} m(\pi_f \otimes \pi_\infty) \mathrm{Tr}\, \pi_\infty(f_\infty),$$

where $m(\pi_f \times \pi_\infty)$ is the multiplicity in

$$\mathrm{L}^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})),$$

and where Π_∞ is the set of adm. reps. of $G(\mathbb{R})$ having trivial central and infin. character.

[Aside: $f_\infty = (-1)^d \cdot$ pseudo. coeff. for some $\pi^0_\infty \in \Pi_\infty$.]

Taking $K_p =$ hyperspec. max. cpt., we recover Kottwitz's theorem.

Simple Shimura varieties

$\mathbb{E}/\mathbb{E}_0/\mathbb{Q}$ CM field, $(D,*)$ central div. alg./$\mathbb{E}$ with positive invol. of second type, G the group defined by

$$G(R) = \{x \in D \otimes_{\mathbb{Q}} R \mid xx^* \in R^\times\}.$$

$X_\infty := G(\mathbb{R}) \cdot h$: in case $\mathbb{E}_0 = \mathbb{Q}$, fix isom. $D \otimes_{\mathbb{Q}} \mathbb{R} \widetilde{=} \mathrm{M}_n(\mathbb{C})$, and $1 \leq r \leq n-1$; we can take

$$h(z) = \mathrm{diag}(z^r, z^{n-r})$$

and $\mu(z) = h_{\mathbb{C}}(z, 1) : \mathbb{C}^\times \to G_{\mathbb{C}}$.

$G(\mathbb{R}) \widetilde{=} \mathrm{GU}(r, n-r)$

Assume: • p inert in $\mathbb{E}_0$, $p = \mathfrak{p}\bar{\mathfrak{p}}$ in $\mathbb{E}$.

• $K = K^p I_p$, $I_p \subset G(\mathbb{Q}_p)$ standard Iwahori.

• (temp.) $G_{\mathbb{Q}_p}$ split ($\widetilde{=} \mathrm{GL}_n \times \mathbb{G}_m$) (and identify $\mu = (1^r, 0^{n-r})$).

Remarks

Let $E = \mathbb{E}_{\mathfrak{p}}$. Then S_K is proper over $\mathcal{O}_E$, smooth over E.

For $r = 1$, used by Harris-Taylor in their proof of the local Langlands correspondence for GL_n.

For general $1 \leq r \leq n-1$, studied by E. Mantovan and also L. Fargues (with arbitrary level structure at p, but less precise results...).

[Aside: deeper level structure always occurs at some prime (if S_K generically smooth), but it is interesting to note that after finite base change, expect tame reduction at such a prime ("potential semi-stable reduction") – in this sense Iwahori level structure is the most "geometric" kind.]

No endoscopy problems (special feature of group G observed by Rapoport-Zink and Kottwitz...)

Key geometric step (part 1))

Identify test function ϕ_j by identifying the function

$$x \mapsto \mathrm{Tr}^{ss}(\Phi^j, \mathrm{R}\Psi_x))$$

on $S_K \times \bar{\mathbb{F}}_p$. We relate to nearby cycles on affine flag variety for $G = \mathrm{GL}_n$ via Rapoport-Zink local models.

Let $k = \mathbb{F}_{p^j}$. Assume $K_p = I_p =$ *Iwahori*.

Theory of Rapoport-Zink: Étale locally, get $S_K \times \bar{k} \cong M^{loc} \times \bar{k}$. Can embed

$$M^{loc} \times \bar{k} \hookrightarrow \mathrm{GL}_n(\bar{k}((t)))/I_{\bar{k},t} := \mathcal{F}L \times \bar{k},$$

the affine flag variety for GL_n.

Any $x \in S_K(\bar{k})$ gives rise to $x_0 \in \mathcal{F}L(\bar{k})$; not unique, but in uniquely determined I-orbit.

So $\mathrm{Tr}^{ss}(\Phi^j, \mathrm{R}\Psi_x) = \mathrm{Tr}^{ss}(\Phi^j, \mathrm{R}\Psi_{x_0}^{M^{loc}})$. Latter is function in Iwahori-Hecke algebra for G. In fact we have the "Kottwitz conjecture":

Theorem 2 (H. and B.C. Ngô; D. Gaitsgory)

$$\mathrm{Tr}^{ss}(\Phi^j, \mathrm{R}\Psi^{M^{loc}}) = q^{d/2} z_{\mu,j},$$

where $z_{\mu,j} \in Z(H(G(\mathbb{Q}_{p^j})//I_j))$ *is the Bernstein function attached to the minuscule dominant cocharacter* μ *of* G.

Here z_{μ_j} is *unique* central function such that $z_{\mu,j} * \mathbb{I}_{K_j} = \mathrm{char}(K_j \mu(p^j) K_j) \in H_{sph}(G(\mathbb{Q}_{p^j}))$. In particular, we know how it acts on unramified principle series...

We proved a more general theorem: for GL of GSp, there is a deformation of affine Grassmannian $\mathcal{G}r_{\mathbb{Q}_p}$ to $\mathcal{F}L_{\mathbb{F}_p}$ such that for $\mathcal{S} \in P_K(\mathcal{G}r)$, nearby cycles $R\Psi(\mathcal{S})$ are "central" in $P_I(\mathcal{F}L)$. Via sheaf-function dictionary, $\mathrm{Tr}^{ss}(\Phi, \mathrm{R}\Psi(\mathcal{S}))$ is *central* function in Iwahori-Hecke algebra.

More about local models

There is a diagram with smooth surjective morphisms

$$S_K \xleftarrow{p} \widetilde{S_K} \xrightarrow{q} M^{loc}.$$

[Definitions: $\widetilde{S_K}$ consists of points $A_\bullet$ together with a rigidification of their de Rham cohomology. M^{loc} consists of chains of $\mathcal{O}_E$ lattices in certain "μ-admissible" relative positions with respect to the standard lattice chain. The morphism q takes the image of the Hodge filtration under the rigidification...]

Definition of M^{loc} for GL_n, $\mu = (1^r, 0^{n-r})$: $M^{loc}(R)$ consists of diagrams

$$\begin{array}{ccccccc} \Lambda_{0,R} & \longrightarrow & \Lambda_{1,R} \cdots & \longrightarrow & \Lambda_{n-1,R} & \xrightarrow{p} & \Lambda_{0,R} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 \cdots & \longrightarrow & \mathcal{F}_{n-1} & \longrightarrow & \mathcal{F}_0; \end{array}$$

$\Lambda_{i,R} := \mathbb{Z}_p\langle p^{-1}e_1, \ldots, p^{-1}e_i, e_{i+1}, \ldots, e_n\rangle \otimes_{\mathbb{Z}_p} R$, and $\mathcal{F}_i$ is an R-submodule, locally a direct summand of rank r.

[Aside (Warning): r, n not the same as before, and really need partial flags version...].

Fundamental lemma (part 2))

Base change homomorphism for central elements in Iwahori-Hecke algebras: Assume G unramified over $\mathbb{Q}_p$; let I_j (resp. I) be the Iwahori subgroups over $\mathbb{Q}_{p^j}$ (resp. $\mathbb{Q}_p$). Let K_j (resp. K) denote corresponding hyperspec. max. cpt. s.g.'s.

Have *Bernstein isomorphism* B:

$$- * \mathbb{I}_K \;:\; Z(H(G//I)) \longrightarrow H_{sph}(G//K)$$

Definition Base change b for $Z(H(G//I))$ is unique map making commute:

$$\begin{array}{ccc} Z(H(G_j//I_j)) & \xrightarrow{B} & H(G_j//K_j) \\ b\downarrow & & \downarrow b \\ Z(H(G//I)) & \xrightarrow{B} & H(G//K), \end{array}$$

where b on right is usual base change for spherical Hecke algebras (defined via Satake isomorphism).

Theorem 3 (H., Ngô) *Let ϕ_j be central in the Iwahori-Hecke algebra over $\mathbb{Q}_{p^j}$. Let $\delta \in G(\mathbb{Q}_{p^j})$ be σ-semi-simple. Then*

$$\mathrm{STO}_{\delta\sigma}(\phi_j) = \mathrm{SO}_{N\delta}(b\phi_j).$$

Also, $\mathrm{SO}_\gamma(b\phi_j) = 0$ *if γ not of form $N\delta$.*

Remarks

In case of spherical Hecke algebras, proved by Clozel and Labesse.

Also get "parahoric version", which is much harder than Iwahori case.

Strategy same as in Labesse. One neat new thing: naive "constant term" $f \mapsto f^{(P)}$ actually preserves central elements (!), and so can use in descent step...

Putting steps 1)-3) together, we proved

Theorem 4 (H.,Ngô) *For S_K simple as above, $\mathrm{Tr}^{ss}(\Phi^j, H^\bullet(S_K \times_E \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell))$ is equal to*

$\mathrm{Tr}(b(z_{\mu,j}) \otimes \mathbb{I}_{K^p} \otimes f_\infty; \mathrm{L}^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})))$.

This implies Theorem 1, since we know how $b(z_{\mu,j})$ acts on $\pi_p^{I_p}$: by the scalar

$$\mathrm{Tr}(r_\mu \phi_{\pi_p}(\Phi \times \begin{bmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{bmatrix}); V_\mu).$$

Towards true local factors

Rapoport: to recover Z from Z^{ss}, enough to know *monodromy-weight conjecture*: The graded gr_k^M of monodromy filtration on $H^i(S_K \times \bar{\mathbb{F}}_p, R\Psi)$ (defined using inertia action) is pure of weight $i + k$.

Nothing known in higher dimensions for situation at hand. However, I can prove cases of *local weight-monodromy conjecture* for the perverse sheaf $R\Psi$.

Automorphic analogue: can recover $L(s, \pi_p, r)$ from $L^{ss}(s, \pi_p, r)$ provided π_p is *tempered*, i.e. local parameter

$$\phi_{\pi_p} : W_{\mathbb{Q}_p} \times \mathsf{SL}_2 \to {}^L G$$

satisfies: $\phi_{\pi_p}(W_{\mathbb{Q}_p})$ is *bounded*.

We expect this for the π_f which come into $H^\bullet(S_K)$.*

*Note: The tempered-ness of the parameter ϕ_{π_p} is *sufficient* for $L^{ss}(s,\pi,r)$ to determine $L(s,\pi,r)$. However, as pointed out by Don Blasius, it is not necessary. Moreover, we actually should not expect all π_p which appear in the cohomology to be tempered.

[In closing, if time: explain

1)heuristic that $(\cdot)^{\Gamma^0}$ necessary for functional equation

2) action of Γ^0 on cohomology of Shimura variety with Iwahori level structure is unipotent (from Gaitsgory's result that action on nearby cycles is unipotent)...]

THE END.