Formulae relating the Bernstein and Iwahori-Matsumoto presentations of an affine Hecke algebra

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Abstract

We consider the "antidominant" variants Θ_λ^- of the elements Θ_λ occurring in the Bernstein presentation of an affine Hecke algebra $\mathcal{H}$. We find explicit formulae for Θ_λ^- in terms of the Iwahori-Matsumoto generators T_w (w ranging over the extended affine Weyl group of the root system R), in the case (i) R is arbitrary and λ is a *minuscule* coweight, or (ii) R is attached to GL_n and $\lambda = me_k$, where e_k is a standard basis vector and $m \geq 1$.

In the above cases, certain *minimal expressions* for Θ_λ^- play a crucial role. Such minimal expressions exist in fact for any coweight λ for GL_n. We give a sheaf-theoretic interpretation of the existence of a minimal expression for Θ_λ^-: the corresponding perverse sheaf on the affine Schubert variety $X(t_\lambda)$ is the push-forward of an explicit perverse sheaf on the Demazure resolution $m : \tilde{X}(t_\lambda) \to X(t_\lambda)$. This approach yields, for a minuscule coweight λ of any R, or for an *arbitrary* coweight λ of GL_n, a conceptual albeit less explicit expression for the coefficient $\Theta_\lambda^-(w)$ of the basis element T_w, in terms of the cohomology of a fiber of the Demazure resolution.

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1 Introduction

Let $\mathcal{H}$ be the affine Hecke algebra associated to a root system. There are two well-known presentations of this algebra by generators and relations, the first discovered by Iwahori-Matsumoto [7] and the second by Bernstein [12], [14]; cf. 2.2.1 below. The Iwahori-Matsumoto presentation reflects the structure of the Iwahori-Hecke algebra $C_c^\infty(I\backslash G/I)$ of the split p-adic group G attached to the root system: the generators T_w correspond to the characteristic functions of Iwahori double cosets IwI, where w ranges over the extended affine Weyl group. The Bernstein presentation reflects the description of the Hecke algebra as an equivariant K-theory of the associated Steinberg variety, which plays a role in the classification of the representations of $\mathcal{H}$, see [9], [2]. The Bernstein presentation has the advantage that one can construct a basis for the center of $\mathcal{H}$ by summing the generators Θ_λ over Weyl-orbits of coweights λ; the resulting functions are known as *Bernstein functions*.

It is of interest to give an explicit relation between the generators in these two presentations. More precisely, one would like to write each Θ_λ as an explicit linear combination of the Iwahori-Matsumoto basis elements T_w. A direct consequence would be the explicit description of the Bernstein functions (and thus the center of $\mathcal{H}$) in terms of the Iwahori-Matsumoto basis. This problem was considered earlier by the first author [4],[5] because of certain applications to the study of Shimura varieties, and was completely answered there for the case where λ is a minuscule coweight. More recently, O. Schiffmann [16] has given explicit formulae for all elements in a certain basis for the center $Z(\mathcal{H})$ of an affine Hecke algebra $\mathcal{H}$ of type A; from this one can derive a formula for the Bernstein function z_μ, where μ is any dominant coweight of a group of type A.

In this paper we consider the "antidominant" variants Θ_λ^- of the elements Θ_λ. The support of these functions is somewhat more regular than the original functions Θ_λ, cf. Lemma 2.1. In section 3 we consider the case where λ is a minuscule coweight, and we prove the following explicit formula for Θ_λ^-.

Theorem 1.1 *Let $\lambda \in X_*$ be minuscule. Then*

$$\Theta_\lambda^- = \sum_{\{x \,:\, \lambda(x)=\lambda\}} \tilde{R}_{x,t_\lambda}(Q)\tilde{T}_x.$$

Here $\lambda(x)$ is the translation part of x "on the left" defined by the decomposition $x = t_{\lambda(x)}w$ $(w \in W_0)$, $\tilde{T}_x$ is a renormalization of the usual Iwahori-Matsumoto generator T_x, $Q = q^{-1/2} - q^{1/2}$, and $\tilde{R}_{x,y}(Q)$ is a variant of the usual R-polynomial of Kazhdan-Lusztig [8].

The formula above is analogous to the expression for Θ_λ found by the first author in Proposition 4.4 of [5]:

$$\Theta_\lambda = \sum_{\{x\,:\,t(x)=\lambda\}} \tilde{R}_{x,t_\lambda}(Q)\tilde{T}_x.$$

Here $t(x)$ is the translation part of x "on the right" defined by the decomposition $x = wt_{t(x)}$ $(w \in W_0)$. However our proof is simpler and more direct than that of loc.cit., and the same arguments appearing here also give a short proof of the formula for Θ_λ. In fact one can derive the formula for Θ_λ^- from that for $\Theta_{-\lambda}$, and vice-versa. Indeed, if $\iota : \mathcal{H} \to \mathcal{H}$ denotes the anti-involution determined by $q^{1/2} \mapsto q^{1/2}$ and $T_x \mapsto T_{x^{-1}}$, then $\iota(\Theta_{-\lambda}) = \Theta_\lambda^-$ and ι interchanges the formulae for $\Theta_{-\lambda}$ and Θ_λ^-. We remark that Theorem 1.1 remains valid for Hecke algebras with arbitrary parameters.

In the fourth section we study coweights of GL_n of the form $\lambda = me_k$, where e_k is the k-th standard basis vector and $m \in \mathbb{Z}_+$. The case $m = 1$, studied in [4] and [5], has relevance to a certain family of Shimura varieties with bad reduction, known as the *Drinfeld case*. The general case is referred to as *multiples of the Drinfeld case*. We prove the following formula for $\Theta_{me_k}^-$.

Theorem 1.2 *Let $1 \le k \le n$ and $m \ge 1$. Then*

$$\Theta_{me_k}^- = \sum_{\{x\,:\,\lambda(x)\preceq me_k\}} \tilde{R}_{x,t_{me_k}}(Q)\tilde{T}_x.$$

Here $\preceq$ denotes the usual partial order on the lattice X_*.

Theorems 1.1 and 1.2 yield explicit expressions for the Bernstein functions z_μ (μ minuscule) and z_{me_1}, respectively; see Corollary 3.6 and 4.2. The expressions in these special cases seem much simpler than the corresponding ones given by Schiffmann [16].

Theorems 1.1 and 1.2 rely on the existence of certain *minimal expressions* for Θ_λ^-: these are expressions of the form

$$\Theta_\lambda^- = \tilde{T}_{t_1}^{\epsilon_1} \cdots \tilde{T}_{t_r}^{\epsilon_r}\tilde{T}_\tau,$$

where $t_\lambda = t_1 \cdots t_r\tau$ $(t_i \in S_a, \tau \in \Omega)$ is a reduced expression and $\epsilon_i \in \{1, -1\}$ for every $1 \le i \le r$. In the final two sections, we discuss how one can approach a general formula for Θ_λ^- when λ is an arbitrary coweight of GL_n, through minimal expressions (which always exist in this setting, cf. section 5). The result is much less explicit than Theorems 1.1 and 1.2, and involves the geometry of the Demazure resolution $\widetilde{X}(t_\lambda) \to X(t_\lambda)$ of the affine Schubert variety $X(t_\lambda)$. We define a perverse sheaf Ξ_λ^- on the affine flag variety whose corresponding function in the Hecke algebra is $\varepsilon_\lambda\Theta_\lambda^-$. It turns out that Ξ_λ^- is supported on $X(t_\lambda)$. We see that the existence of a minimal expression for Θ_λ^- is analogous to the existence of a certain explicitly determined perverse sheaf on $\widetilde{X}(t_\lambda)$ whose push-forward to $X(t_\lambda)$ is Ξ_λ^-. More precisely, we conclude the paper with the following result (cf. Theorem 6.7, Corollary 6.8 for a completely precise statement).

Theorem 1.3 *Let λ be a minuscule coweight of a root system, or any coweight for GL_n. Choose a minimal expression for Θ_λ^- and let $m : \widetilde{X}(t_\lambda) \to X(t_\lambda)$ denote the corresponding Demazure resolution for $X(t_\lambda)$. Then there exists an explicit perverse sheaf $\mathcal{D}$ on $\widetilde{X}(t_\lambda)$ (determined by the choice of minimal expression for Θ_λ^-) such that*

$$Rm_*(\mathcal{D}) = \Xi_\lambda^-.$$

Consequently, if we denote the coefficient of T_x in the expression for Θ_λ^- by $\Theta_\lambda^-(x)$, then we have

$$\Theta_\lambda^-(x) = \varepsilon_\lambda\mathrm{Tr}(\mathrm{Fr}_q, H^\bullet(m^{-1}(x), \mathcal{D})),$$

for any $x \le t_\lambda$ in the Bruhat order. Here the right hand side denotes the alternating trace of Frobenius on the étale cohomology of the fiber over $x \in X(t_\lambda)$ with coefficents in the sheaf $\mathcal{D}$.

Given a coweight λ for an arbitrary root system, let λ_d denote the dominant coweight in its Weyl-orbit. We remark that there is a similar formula for Θ_λ^-, *provided that λ_d is a sum of minuscule dominant coweights.*

2 Preliminaries

2.1 Affine Weyl group

Let $(X^*, X_*, R, \check{R}, \Pi)$ be a root system, where Π is the set of simple roots. The Weyl group W_0 is generated by the set of simple reflections $\{s_\alpha : \alpha \in \Pi\}$.

We define a partial order $\preceq$ on X_* (resp. X^*) by setting $\lambda \preceq \mu$ whenever $\mu - \lambda$ is a linear combination with nonnegative integer coefficients of elements of $\{\check{\alpha} : \alpha \in \Pi\}$ (resp. $\{\alpha : \alpha \in \Pi\}$). We let Π_m denote the set of roots $\beta \in R$ such that β is a minimal element of $R \subset X^*$ with respect to $\preceq$.

In section 4 we will use the following description of the relation $\preceq$ for coweights of GL_n: $(\lambda_1, \ldots, \lambda_n) \preceq (\mu_1, \ldots \mu_n)$ if and only if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $1 \leq i \leq n-1$, and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$.

Let $\widetilde{W}$ be the semidirect product $X_* \rtimes W_0 = \{t_x w : w \in W_0, x \in X_*\}$, with multiplication given by $t_x w t_{x'} w' = t_{x+w(x')} ww'$. For any $x \in \widetilde{W}$, there exists a unique expression $t_{\lambda(x)} w$, where $w \in W_0$ and $\lambda(x) \in X_*$.

Let
$$S_a = \{s_\alpha : \alpha \in \Pi\} \cup \{t_{-\check{\alpha}} s_\alpha : \alpha \in \Pi_m\} \subset \widetilde{W}.$$

Define length $l : \widetilde{W} \to \mathbb{Z}$ by

$$l(t_x w) = \sum_{\alpha \in R^+ : w^{-1}(\alpha) \in R^-} |\langle \alpha, x \rangle - 1| + \sum_{\alpha \in R^+ : w^{-1}(\alpha) \in R^+} |\langle \alpha, x \rangle|.$$

Let $\check{Q}$ be the subgroup of X_* generated by $\check{R}$. The subgroup $W_a = \check{Q} \rtimes W_0$ of $\widetilde{W}$ is a Coxeter group with S_a the set of simple reflections. The subgroup is normal and admits a complement $\Omega = \{w \in \widetilde{W} : l(w) = 0\}$.

For $w \in \widetilde{W}$ denote $\varepsilon_w = (-1)^{l(w)}$ and $q_w = q^{l(w)}$ (for q any parameter).

The Coxeter group (W_a, S_a) comes equipped with the Bruhat order $\leq$. We extend it to $\widetilde{W}$ as follows: we say $w\tau \leq w'\tau'$ ($w, w' \in W_a$, $\tau, \tau' \in \Omega$) if $w \leq w'$ and $\tau = \tau'$.

Let $\mu \in X_*$ be dominant. Following Kottwitz-Rapoport [10], we say $x \in \widetilde{W}$ is μ-*admissible* if $x \leq t_{w(\mu)}$ for some $w \in W_0$. We denote the set of μ-admissible elements by $\mathrm{Adm}(\mu)$.

2.2 Hecke algebra

2.2.1 Presentations

The braid group of $\widetilde{W}$ is the group generated by T_w ($w \in \widetilde{W}$) with relations

$$T_w T_{w'} = T_{ww'} \qquad \text{whenever } l(ww') = l(w) + l(w').$$

The Hecke algebra $\mathcal{H}$ is defined to be the quotient of the group algebra (over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$) of the braid group of $\widetilde{W}$, by the two-sided ideal generated by the elements

$$(T_s + 1)(T_s - q),$$

for $s \in S_a$. The image of T_w in $\mathcal{H}$ is again denoted by T_w. It is known that the elements T_w ($w \in \widetilde{W}$) form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for $\mathcal{H}$. The presentation of $\mathcal{H}$ using the generators T_w and the above relations is called the *Iwahori-Matsumoto presentation*.

For any T_w, define a renormalization $\tilde{T}_w = q^{-l(w)/2} T_w$. Define an indeterminate $Q = q^{-1/2} - q^{1/2}$. The elements $\tilde{T}_w$ form a basis for $\mathcal{H}$, and the usual relations can be written as

$$\tilde{T}_s \tilde{T}_w = \begin{cases} \tilde{T}_{sw}, & \text{if } l(sw) = l(w) + 1, \\ -Q\tilde{T}_w + \tilde{T}_{sw}, & \text{if } l(sw) = l(w) - 1, \end{cases}$$

for $w \in \widetilde{W}$ and $s \in S_a$. There is also a right-handed version of this relation. Note that $\tilde{T}_s^{-1} = \tilde{T}_s + Q$.

We will denote $\tilde{T}_{t_\lambda}$ ($\lambda \in X_*$) simply by $\tilde{T}_\lambda$.

For $\lambda \in X_*$, define

$$\Theta_\lambda = \tilde{T}_{\lambda_1} \tilde{T}_{\lambda_2}^{-1}$$

where $\lambda = \lambda_1 - \lambda_2$, and λ_1, λ_2 are dominant. The elements Θ_λ generate a commutative subalgebra of $\mathcal{H}$. It is known that the elements $\Theta_\lambda T_w$ ($\lambda \in X_*$, $w \in W_0$) form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for $\mathcal{H}$. These generators satisfy well-known relations (see Prop. 3.6, [14]); in case the root system is simply connected, these are given by the formula

$$\Theta_\lambda T_s - T_s \Theta_{s(\lambda)} = (q-1) \frac{\Theta_\lambda - \Theta_{s(\lambda)}}{1 - \Theta_{-\check\alpha}},$$

where $s = s_\alpha$ and $\alpha \in \Pi$. The presentation of $\mathcal{H}$ with generators $\Theta_\lambda T_w$ and the above relations is called the *Bernstein presentation*.

We also define

$$\Theta_\lambda^- = \tilde{T}_{\lambda_1'} \tilde{T}_{\lambda_2'}^{-1}$$

where $\lambda = \lambda_1' - \lambda_2'$, and λ_1', λ_2' are *antidominant*.

The involution $a \to \bar{a}$ of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ determined by $q \mapsto q^{-1}$ extends to an involution $h \to \bar{h}$, given by

$$\overline{\sum a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1}.$$

It is immediate that $\overline{\Theta}_\lambda = \Theta_\lambda^-$. Clearly the Bernstein presentation gives rise to an analogous presentation using the generators $\Theta_\lambda^- T_w$ in place of $\Theta_\lambda T_w$.

2.2.2 Bernstein functions

For each W_0-orbit M in X_*, define the Bernstein function z_M attached to M by

$$z_M = \sum_{\lambda \in M} \Theta_\lambda.$$

When the W_0-orbit M contains the dominant element μ, this function is denoted by z_μ.

From Corollary 8.8 of Lusztig [12], we have $z_\mu = \bar{z}_\mu$. Consequently,

$$z_\mu = \sum_{\lambda \in W_0(\mu)} \Theta_\lambda^-.$$

2.2.3 A support property

The preceding formula implies that when one studies Bernstein functions there is no harm in working with the functions Θ_λ^- instead of the functions Θ_λ. We do so in this paper because their supports enjoy a nice regularity property, given by the following lemma.

Lemma 2.1 *For $\lambda \in X_*$, we have*

$$\text{supp}(\Theta_\lambda^-) \subset \{x \ : \ \lambda(x) \preceq \lambda\}.$$

Proof. Write

$$\Theta_\lambda^- = \sum_{y \leq t_\lambda} a_y(Q) \tilde{T}_y,$$

where $a_y(Q) \in \mathbb{Z}_+[Q]$ (see Lemma 5.1 and Corollary 5.7 of [4]).

Choose a dominant coweight μ' such that $\mu' + \lambda(x)$ is also dominant for any x in the support of Θ_λ^-. Thus we have

$$\tilde{T}_{-(\mu'+\lambda)}^{-1} = \Theta_{\mu'+\lambda}^- = \Theta_{\mu'}^- \Theta_\lambda^- = \sum_y a_y(Q) \tilde{T}_{-\mu'}^{-1} \tilde{T}_y.$$

Let $y \in \text{supp}(\Theta_\lambda^-)$. We claim that $t_{\mu'} y$ belongs to the support of $\tilde{T}_{-\mu'}^{-1} \tilde{T}_y$. Indeed, under the specialization map $\mathcal{H} \to \mathbb{Z}[\widetilde{W}]$ determined by $q^{1/2} \mapsto 1$, the element $\tilde{T}_{-\mu'}^{-1} \tilde{T}_y$ maps to $t_{\mu'} y$. Since no cancellation occurs on the right hand side above, we see from this that $t_{\mu'} y \in \text{supp}(\tilde{T}_{-(\mu'+\lambda)}^{-1})$, and thus

$$t_{\mu'+\lambda(y)} w_y = t_{\mu'} y \leq t_{\mu'+\lambda},$$

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where $y = t_{\lambda(y)}w_y$. Since $\mu' + \lambda(y)$ and $\mu' + \lambda$ are both dominant, it is well-known that this implies $\mu' + \lambda(y) \preceq \mu' + \lambda$. The lemma follows. ∎

In the case where λ is minuscule, this statement can be considerably sharpened; see Corollary 3.5. We remark that Lemma 2.1 plays a key role in the proof of Theorem 4.1.

2.2.4   R-polynomials

For any $y \in \widetilde{W}$, let $y = s_1 \cdots s_r \tau$ ($s_i \in S_a, \tau \in \Omega$) be a reduced expression for y. Then for any x, we can write

$$\tilde{T}_{y^{-1}}^{-1} = \sum_{x \in \widetilde{W}} \widetilde{R}_{x,y}(Q)\tilde{T}_x.$$

where $\widetilde{R}_{x,y}(Q) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. These coefficients $\widetilde{R}_{x,y}(Q)$ can be thought of as polynomial expressions in Q (as the notation suggests) because of the identity

$$\tilde{T}_{y^{-1}}^{-1} = (\tilde{T}_{s_1} + Q) \cdots (\tilde{T}_{s_r} + Q)\tilde{T}_\tau.$$

3   The minuscule case

We say $\lambda \in X_*$ is *minuscule* if $\langle \alpha, \lambda \rangle \in \{0, \pm 1\}$, for every root $\alpha \in R$. Such coweights are the concern of this section.

The purpose of this section is to present an analogue of Proposition 4.4 from [5] using Θ_λ^- instead of Θ_λ. For simplicity, the theorem is given here for affine Hecke algebras with trivial parameter systems. The generalization to arbitrary parameter systems is straightforward (see [5] for notation and details). Similar arguments to those appearing here apply to Θ_λ, giving a short proof of Proposition 4.4 from [5].

Theorem 3.1 *Let μ^- be minuscule and antidominant, and $\lambda \in W_0(\mu^-)$. Then*

$$\Theta_\lambda^- = \sum_{x \,:\, \lambda(x) = \lambda} \widetilde{R}_{x,t_\lambda}(Q)\tilde{T}_x.$$

We begin with some lemmas. For a proof of the first lemma, refer to Proposition 3.4 of [5], where a similar result is given (see also the proof of Corollary 6.6).

Lemma 3.2 *Let μ^- be an antidominant and minuscule coweight, and let $\tau \in \Omega$ be the unique element such that $t_{\mu^-} \in W_a\tau$. Let $\lambda \in W_0(\mu^-)$. Suppose that $\lambda - \mu^-$ is a sum of p simple coroots ($0 \le p \le l(t_{\mu^-}) = r$). Then there exists a sequence of simple roots $\alpha_1, \ldots, \alpha_p$ such that the following hold (setting $s_i = s_{\alpha_i}$):*
(1) $\langle \alpha_i, s_{i-1} \cdots s_1(\mu^-) \rangle = -1, \forall 1 \le i \le p$;
(2) there is a reduced expression for t_{μ^-} of the form $t_{\mu^-} = s_1 \cdots s_p t_1 \cdots t_{r-p}\tau$;
(3) there is a reduced expression for t_λ of the form $t_\lambda = t_1 \cdots t_{r-p}(^\tau s_1) \cdots (^\tau s_p)\tau$;
(4) $\Theta_\lambda^- = \tilde{T}_{t_1} \cdots \tilde{T}_{t_{r-p}} \tilde{T}_{\tau s_1}^{-1} \cdots \tilde{T}_{\tau s_p}^{-1} \tilde{T}_\tau$,
where $t_j \in S_a, \forall j \in \{1, \ldots, r-p\}$.

Lemma 3.3 *Let $x \in \widetilde{W}$, and suppose that $xs_\alpha > x$ for all $\alpha \in \Pi$. Then $l(xw) = l(x) + l(w)$ for all $w \in W_0$.*

Since μ^- is antidominant, this lemma applies to the expression $t_{\mu^-} = s_1 \cdots s_p t_1 \cdots t_{r-p}\tau$. It then applies to the expression $t_1 \cdots t_{r-p}\tau$ as well. It follows that we can think of the formula in Lemma 3.2(4) as

$$\Theta_\lambda^- = \tilde{T}_{w^\lambda}\tilde{T}_{s_1}^{-1} \cdots \tilde{T}_{s_p}^{-1} = \tilde{T}_{w^\lambda}\tilde{T}_{w^{-1}}^{-1}$$

where $t_\lambda = w^\lambda w$, with $w \in W_0$ and w^λ the minimal length representative for the coset $t_\lambda W_0$. This observation is helpful towards proving the main result of this section.

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Lemma 3.4 *For λ, $s_1, \ldots, s_p$, and $t_1, \ldots, t_{r-p}$ as in Lemma 3.2, the mapping*

$$\{y \ : \ y \leq \ s_1 \cdots s_p\} \ \longrightarrow \ \{x \ : \ \lambda(x) = \lambda \text{ and } x \leq t_\lambda\}$$

defined by

$$y \ \mapsto \ t_1 \cdots t_{r-p} \tau y = w^\lambda y$$

is bijective.

Proof of Theorem 3.1. Let $w = s_1 \cdots s_p$, and $w^\lambda = t_1 \cdots t_{r-p} \tau$, so that $t_{\mu^-} = w w^\lambda$ and $t_\lambda = w^\lambda w$. We have

$$\tilde{T}_{s_1}^{-1} \cdots \tilde{T}_{s_p}^{-1} = \sum_{y \ : \ y \leq w} \widetilde{R}_{y,w}(Q) \tilde{T}_y$$

The expression for Θ_λ^- of Lemma 3.2, together with the fact that $\tilde{T}_{w^\lambda} \tilde{T}_y = \tilde{T}_{w^\lambda y}$ for all $y \in W_0$ (since $l(w^\lambda y) = l(w^\lambda) + l(y)$), implies

$$\Theta_\lambda^- = \sum_{y \ : \ y \leq w} \widetilde{R}_{y,w}(Q) \tilde{T}_{w^\lambda y}.$$

Using the recursion formula of Lemma 2.5 (1) from [5], we obtain $\widetilde{R}_{y,w}(Q) = \widetilde{R}_{w^\lambda y, t_\lambda}(Q)$. In view of the bijection given in Lemma 3.4, we have

$$\sum_{y \ : \ y \leq w} \widetilde{R}_{w^\lambda y, t_\lambda}(Q) \tilde{T}_{w^\lambda y} = \sum_{x \ : \ \lambda(x)=\lambda} \widetilde{R}_{x,t_\lambda}(Q) \tilde{T}_x,$$

which completes the proof. ∎

For the minuscule case, Theorem 3.1 yields the following improvement on Lemma 2.1.

Corollary 3.5 *Let $\lambda \in X_*$ be minuscule. Then*

$$\mathrm{supp}(\Theta_\lambda^-) = \{x \ : \ \lambda(x) = \lambda \text{ and } x \leq t_\lambda\}.$$

Here we have used Lemma 2.5 (5) of [5], which asserts that $\tilde{R}_{x,y}(Q) \neq 0$ if and only if $x \leq y$.

The Bernstein function z_μ has a very simple form when μ is minuscule (cf. Theorem 4.3 of [5]):

Corollary 3.6 *If μ is dominant and minuscule, then*

$$z_\mu = \sum_{x \in \mathrm{Adm}(\mu)} \tilde{R}_{x,t_{\lambda(x)}}(Q) \tilde{T}_x.$$

4   Multiples of the Drinfeld case

Fix positive integers n and m, and an integer $1 \leq k \leq n$. In this section, we establish in Theorem 4.1 a formula for the Θ_λ^- functions of GL_n when $\lambda = m e_k$ (where e_k denotes the coweight of GL_n with kth coordinate equal to 1, and all other coordinates equal to 0).

In this section, we adopt the following notation: for $1 \leq i \leq n - 1$, let $\alpha_i = \check{\alpha}_i = e_i - e_{i+1}$, and let $s_i = s_{\alpha_i}$. We single out the element $\tau \in \Omega$ given by $\tau = t_{(1,0,\ldots,0)} s_1 \cdots s_{n-1}$.

Theorem 4.1 *For the coweight $m e_k$ of GL_n, we have*

$$\Theta_{m e_k}^- = \sum_{x \ : \ \lambda(x) \preceq m e_k} \widetilde{R}_{x,t_{m e_k}}(Q) \tilde{T}_x.$$

Consequently, we have a result for me_k analogous to that of Corollary 3.5 for λ minuscule, that is,

$$\text{supp}(\Theta^-_{me_k}) = \{x \ : \ \lambda(x) \preceq me_k \text{ and } x \le t_{me_k}\}.$$

We also get the following explicit formula for the Bernstein function z_{me_1}, analogous to Corollary 3.6:

Corollary 4.2

$$z_{me_1} = \sum_{x \in \text{Adm}(\mu)} \left(\sum_{\lambda(x) \preceq \lambda \ , \ x \le t_\lambda} \tilde{R}_{x,t_\lambda}(Q) \right) \tilde{T}_x,$$

where the inner sum ranges over $\lambda \in W_0(me_1)$ such that $\lambda(x) \preceq \lambda$ and $x \le t_\lambda$.

We require three lemmas before the proof of the theorem. In the following arguments we use the notation $\prod$ to denote products even though we are working in a non-commutative ring. We will use the following convention: $\prod_{i=1}^n a_i$ will denote the product $a_1 a_2 \cdots a_n$ (in that order).

Lemma 4.3 *For the coweight me_k of* GL_n*, we have*

$$\Theta^-_{me_k} = (\tilde{T}_{s_{k-1}} \cdots \tilde{T}_{s_1} \tilde{T}_\tau \tilde{T}^{-1}_{s_{n-1}} \cdots \tilde{T}^{-1}_{s_k})^m.$$

Proof. From Lemma 3.2, we have

$$\Theta^-_{e_k} = \tilde{T}_{s_{k-1}} \cdots \tilde{T}_{s_1} \tilde{T}_\tau \tilde{T}^{-1}_{s_{n-1}} \cdots \tilde{T}^{-1}_{s_k}.$$

Then the formula for $\Theta^-_{me_k}$ follows from the fact that $\Theta^-_{\nu_1 + \nu_2} = \Theta^-_{\nu_1} \Theta^-_{\nu_2}$ for all $\nu_1, \nu_2 \in X_*$. ∎

Lemma 4.4 *Let $w, y \in \tilde{W}$. Then*

$$\tilde{T}_w \tilde{T}_y = \sum_{x = \tilde{w}\tilde{y}} a_x \tilde{T}_x$$

where $\tilde{w}$ and $\tilde{y}$ range over certain subexpressions of w and y, respectively.

Proof. This is an easy induction on the length of y. ∎

Lemma 4.5 *Let x be a subexpression of*

$$t_{me_k} = (s_{k-1} \cdots s_1 \tau s_{n-1} \cdots s_k)^m$$

such that for some $1 \le i \le k-1$, at least one s_i is deleted. Then $\lambda(x) \not\preceq me_k$.

Proof. We can write

$$x = u_1 \tau v_1 \cdots u_m \tau v_m$$

for suitable subexpressions $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ of $s_{k-1} \cdots s_1$ and $s_{n-1} \cdots s_k$, respectively. Suppose that p is the least index such that $u_p \ne s_{k-1} \cdots s_1$. Then

$$x = \left(\prod_{i=1}^{p-1} t_{e_k} s_k \cdots s_{n-1} v_i \right) (t_{e_j} u_p s_1 \cdots s_{n-1} v_p) \left(\prod_{i=p+1}^m u_i \tau v_i \right),$$

for some j with $1 \le j < k$. Since $s_k \cdots s_{n-1} v(e_j) = e_j$ for any subexpression v of $s_{n-1} \cdots s_k$,

$$\left(\prod_{i=1}^{p-1} t_{e_k} s_k \cdots s_{n-1} v_i \right) (t_{e_j}) \left(\prod_{i=1}^{p-1} t_{e_k} s_k \cdots s_{n-1} v_i \right)^{-1} = t_{e_j}.$$

It follows that the translation part $\lambda(x)$ is the sum of e_j and a non-negative integral linear combination of vectors e_i $(i = 1, \ldots, n)$. Indeed, the translation part of

$$\left(\prod_{i=1}^{p-1} t_{e_k} s_k \cdots s_{n-1} v_i \right) (u_p s_1 \cdots s_{n-1} v_p) \left(\prod_{i=p+1}^{m} u_i \tau v_i \right)$$

is necessarily a vector $(b_1, \ldots, b_n)$ where $b_i \in \mathbb{Z}_+$ for every i.

We thus see that one of the first $k-1$ coordinates of $\lambda(x)$ is positive (namely the j-th coordinate is), and this implies that $\lambda(x) \npreceq m e_k$. ∎

Proof of Theorem 4.1. Let $E = (\{0,1\}^{n-1})^m - (\{1\}^{k-1} \times \{0,1\}^{n-k})^m$. Using $\tilde{T}_s^{-1} = \tilde{T}_s + Q$ and expanding the left hand side, we can write

$$\tilde{T}_{-m e_k}^{-1} = \Theta_{m e_k}^- + \sum_{\epsilon \in E} Q^{m(n-1) - \sigma(\epsilon)} \prod_{i=1}^{m} \tilde{T}_{s_{k-1}}^{\epsilon_{k-1}^i} \cdots \tilde{T}_{s_1}^{\epsilon_1^i} \tilde{T}_\tau \tilde{T}_{s_{n-1}}^{\epsilon_{n-1}^i} \cdots \tilde{T}_{s_k}^{\epsilon_k^i}. \tag{1}$$

Here $\epsilon_j^i \in \{0,1\}$ and we denote $\epsilon = (\epsilon^1, \ldots, \epsilon^m)$, and $\epsilon^i = (\epsilon_1^i, \ldots, \epsilon_{n-1}^i)$, $i = 1, \ldots, m$, and $\sigma(\epsilon) = \sum_{i=1}^{m} \sum_{j=1}^{n-1} \epsilon_j^i$.

From Lemma 2.1, we know that $\mathrm{supp}(\Theta_{m e_k}^-) \subset \{x : \lambda(x) \preceq m e_k\}$. Thus we need only prove that if $\tilde{T}_x$ is in the support of the second term on the right hand side, then $\lambda(x) \npreceq m e_k$. Indeed, then the first and second terms on the right hand side of (1) have disjoint supports, and so the coefficients of like terms will be equal in $\tilde{T}_{-m e_k}^{-1}$ and $\Theta_{m e_k}^-$.

Let $\epsilon = (\epsilon^1, \ldots, \epsilon^m) \in E$, and consider

$$\prod_{i=1}^{m} \tilde{T}_{s_{k-1}}^{\epsilon_{k-1}^i} \cdots \tilde{T}_{s_1}^{\epsilon_1^i} \tilde{T}_\tau \tilde{T}_{s_{n-1}}^{\epsilon_{n-1}^i} \cdots \tilde{T}_{s_k}^{\epsilon_k^i}.$$

By Lemma 4.4, if x is in the support of this product, then x is a subexpression of

$$\prod_{i=1}^{m} s_{k-1}^{\epsilon_{k-1}^i} \cdots s_1^{\epsilon_1^i} \tau s_{n-1}^{\epsilon_{n-1}^i} \cdots s_k^{\epsilon_k^i}.$$

Since E excludes the elements of $(\{1\}^{k-1} \times \{0,1\}^{n-k})^m$, we know that for some $1 \le i \le m$ and $1 \le j \le k-1$, that $\epsilon_j^i = 0$. But this is equivalent to the deletion of some s_j $(1 \le j \le k-1)$ from the expression $t_{m e_k} = (s_{k-1} \cdots s_1 \tau s_{n-1} \cdots s_k)^m$. By Lemma 4.5, we have $\lambda(x) \npreceq m e_k$, and the proof is complete. ∎

5 Minimal expressions

We say Θ_λ^- has a *minimal expression* if it can be written in the form

$$\Theta_\lambda^- = \tilde{T}_{t_1}^{\epsilon_1} \cdots \tilde{T}_{t_r}^{\epsilon_r} \tilde{T}_\tau,$$

where $t_\lambda = t_1 \cdots t_r \tau$ $(t_i \in S_a, \tau \in \Omega)$ is a reduced expression and $\epsilon_i \in \{1, -1\}$ for every $1 \le i \le r$. Such expressions played a key role in Theorems 3.1 and 4.1.

Lemma 3.2 asserts that Θ_λ^- has a minimal expression whenever λ is minuscule. If λ is any coweight for GL_n, then we may write

$$\lambda = \lambda_1 + \cdots + \lambda_k$$

where each λ_j is minuscule and

$$l(t_\lambda) = l(t_{\lambda_1}) + \cdots + l(t_{\lambda_k}).$$

It follows that for any coweight λ of GL_n, there is a minimal expression for Θ_λ^-. Letting w^{λ_i} denote the minimal representative for the coset $t_{\lambda_i} W_0$ and writing $t_{\lambda_i} = w^{\lambda_i} w_i$ ($w_i \in W_0$), we may recover a minimal expression for

$$\Theta_\lambda^- = \tilde{T}_{w^{\lambda_1}} \tilde{T}_{w_1^{-1}}^{-1} \cdots \tilde{T}_{w^{\lambda_k}} \tilde{T}_{w_k^{-1}}^{-1}$$

by choosing reduced expressions for every w^{λ_i} and w_i.

Clearly a similar result would follow for any root system with the property that Θ_ω^- has a minimal expression for every Weyl conjugate ω of every fundamental coweight. It seems to be an interesting combinatorial problem to determine the root systems (besides that for GL_n) which satisfy this property.

In principle, a minimal expression for Θ_λ^- allows one to write it as an explicit linear combination of the Iwahori-Matsumoto generators T_w, simply by using the formula $\tilde{T}_s^{-1} = \tilde{T}_s + Q$ and expanding the product. The result is a linear combination of certain products

$$T_{s_1} \cdots T_{s_g} T_\sigma$$

($s_i \in S_a$, $\sigma \in \Omega$), where $s_1 \cdots s_g \sigma$ ranges over certain subexpressions of t_λ (which subexpressions occur is governed by the signs ϵ_j in the minimal expression). These may in turn be simplified by using the well-known formula

$$T_{s_1} \cdots T_{s_g} = \sum_w N(\underline{s}, w, q) T_w,$$

(cf. [11], Lemma 3.7). Here $\underline{s} = (s_1, \ldots, s_g)$ and $N(\underline{s}, w, q)$ is the number of $\mathbb{F}_q$-rational points on the variety $Z(\underline{s}, w)$ consisting of all sequences $(I_1, \ldots, I_g)$ where the I_i are Iwahori subgroups of $G^{sc}(\bar{\mathbb{F}}_q(\!(t)\!))$ (here G^{sc} is the simply connected group associated to the given root system) such that the relative positions of adjacent subgroups satisfy

$$\mathrm{inv}(I_{i-1}, I_i) = s_i$$

for all $1 \leq i \leq g$, and $I_g = w I_0 w^{-1}$, where I_0 is a fixed "standard" Iwahori subgroup.

We forgo the cumbersome task of describing more completely the resulting expressions for Θ_λ^- in terms of the generators T_w. The combinatorics are best described in the geometric framework of Demazure resolutions. We explain this in the following section.

Remark. Let λ be a coweight for GL_n, and write $\lambda = m_1 e_1 + \cdots + m_n e_n$. One finds a similar expression for Θ_λ^- starting from

$$\Theta_\lambda^- = \Theta_{m_1 e_1}^- \cdots \Theta_{m_n e_n}^-,$$

and making use of Theorem 4.1.

6 Sheaf-theoretic meaning of minimal expressions

The goal of this section is to describe a sheaf-theoretic interpretation of a minimal expression for Θ_λ^-: the corresponding perverse sheaf on the affine flag variety is the push-forward of an explicit perverse sheaf on a Demazure resolution of the Schubert variety $X(t_\lambda)$. We proceed to illustrate this statement in more detail.

6.1 Affine flag variety

Let $k = \mathbb{F}_q$ denote the finite field with q elements, and let $\bar{k}$ denote an algebraic closure of k. Let G be the split connected reductive group over k whose root system is $(X^*, X_*, R, \check{R}, \Pi)$. Choose a split torus T and a k-rational Borel subgroup B containing T, which give rise to R and Π.

Denote by $\mathcal{F}l$ the affine flag variety for G. This is an ind-scheme over k whose k-points are given by

$$\mathcal{F}l(k) = G(k(\!(t)\!))/I_k,$$

where $I = I_k \subset G(k[[t]])$ is the Iwahori subgroup whose reduction modulo t is B.

Fix a prime $\ell \neq \mathrm{char}(k)$, and make a fixed choice for $\sqrt{q} \in \bar{\mathbb{Q}}_\ell$ (for Tate twists).

Let $D^b(\mathcal{F}l)$ denote the category $D_c^b(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$. By definition $D_c^b(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$ is the inductive 2-limit of categories $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ where $X \subset \mathcal{F}l$ ranges over all projective k-schemes which are closed subfunctors of the ind-scheme

$\mathcal{F}l$. The category $D_c^b(X, \mathbb{Q}_\ell)$ is the "derived" category of Deligne [3]: $\mathbb{Q}_\ell \otimes$ the projective 2-limit of the categories $D_{ctf}^b(X, \mathbb{Z}/\ell^n\mathbb{Z})$. For any finite extension E of $\mathbb{Q}_\ell$ contained in $\bar{\mathbb{Q}}_\ell$, the definition of $D_c^b(X, E)$ is similar, and by definition $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ is the inductive 2-limit of the categories $D_c^b(X, E)$.

For $f : X \to Y$ a morphism of finite-type k-schemes, we have the four "derived" functors $f_*, f_! : D_c^b(X, \bar{\mathbb{Q}}_\ell) \to D_c^b(Y, \bar{\mathbb{Q}}_\ell)$ and $f^*, f^! : D_c^b(Y, \bar{\mathbb{Q}}_\ell) \to D_c^b(X, \bar{\mathbb{Q}}_\ell)$. This notation should cause no confusion, since we never use the non-derived versions of the pull-back and push-forward functors in this paper.

We define the category $P_I(\mathcal{F}l)$: it is the full subcategory of $D^b(\mathcal{F}l)$ whose objects are I-equivariant perverse sheaves for the middle perversity (by definition the latter have finite dimensional support).

The I-orbits on $\mathcal{F}l$ correspond to $\widetilde{W}$. Given $w \in \widetilde{W}$, we denote by $Y(w) = IwI/I$ the corresponding Bruhat cell, and we denote its closure by $X(w) = \overline{Y(w)}$. Further, let $\bar{\mathbb{Q}}_{\ell,w}$ denote the constant sheaf on $Y(w)$, and define $\mathcal{A}_w = \bar{\mathbb{Q}}_{\ell,w}[l(w)](l(w)/2)$. This is a self-dual perverse sheaf on $Y(w)$.

Let $j_w : Y(w) \hookrightarrow X(w)$ denote the open immersion. We define $J_{w*} = j_{w*}\mathcal{A}_w$ and $J_{w!} = j_{w!}\mathcal{A}_w$. These are perverse sheaves in $P_I(\mathcal{F}l)$ satisfying $D(J_{w*}) = J_{w!}$. (Here D denotes Verdier duality.)

Given $\mathcal{G} \in P_I(\mathcal{F}l)$ we may define the corresponding function $[\mathcal{G}]$ on $\mathcal{F}l(k)$, which we may identify with an element in $\mathcal{H}$:

$$[\mathcal{G}](x) = \text{Tr}(\text{Fr}_q, \mathcal{G}_x),$$

where Fr_q denotes the Frobenius morphism on $\mathcal{F}l_{\bar{k}}$ (raising coordinates to power q). We have

$$[J_{w!}] = \varepsilon_w q_w^{-1/2} T_w$$
$$[J_{w*}] = \varepsilon_w q_w^{1/2} T_{w^{-1}}^{-1}.$$

6.2 Convolution of sheaves

Following Lusztig [15], one can define a convolution product $\star : P_I(\mathcal{F}l) \times P_I(\mathcal{F}l) \to D^b(\mathcal{F}l)$. We formulate this in a way similar to [6]. Given $\mathcal{G}_i \in P_I(\mathcal{F}l)$, $i = 1, 2$, we can choose $X(w_i)$ such that the support of $\mathcal{G}_i$ is contained in $X(w_i)$, for $i = 1, 2$. We may identify $\mathcal{F}l$ with the space of all "affine flags" $\mathcal{L}$ for $G(k((t)))$; there is a base point $\mathcal{L}_0$ whose stabilizer in $G(k((t)))$ is the "standard" Iwahori subgroup I. Then $X(w)$ is identified with the space of all affine flags $\mathcal{L}$ such that the relative position between the base point $\mathcal{L}_0$ and $\mathcal{L}$ satisfies

$$\text{inv}(\mathcal{L}_0, \mathcal{L}) \leq w$$

in the Bruhat order on $\widetilde{W}$. The "twisted" product $X(w_1)\tilde{\times}X(w_2)$ is the space of pairs $(\mathcal{L}, \mathcal{L}') \in \mathcal{F}l \times \mathcal{F}l$ such that

$$\text{inv}(\mathcal{L}_0, \mathcal{L}) \leq w_1$$
$$\text{inv}(\mathcal{L}, \mathcal{L}') \leq w_2.$$

We can find a finite-dimensional projective subvariety $X \subset \mathcal{F}l$ with the property that $(\mathcal{L}, \mathcal{L}') \in X(w_1)\tilde{\times}X(w_2) \Rightarrow \mathcal{L}' \in X$. The "multiplication" map $m : X(w_1)\tilde{\times}X(w_2) \to X$ given by $(\mathcal{L}, \mathcal{L}') \mapsto \mathcal{L}'$ is proper.

Now $\mathcal{G}_i$ $(i = 1, 2)$ determine a well-defined perverse sheaf $\mathcal{G}_1\tilde{\boxtimes}\mathcal{G}_2$ on $X(w_1)\tilde{\times}X(w_2)$ (see e.g. [6]). We define

$$\mathcal{G}_1 \star \mathcal{G}_2 = m_*(\mathcal{G}_1\tilde{\boxtimes}\mathcal{G}_2).$$

The convolution $\mathcal{G}_1 \star \mathcal{G}_2 \in D^b(\mathcal{F}l)$ is independent of the choice of $X(w_i)$ and X.

The object $\mathcal{G}_1 \star \mathcal{G}_2$ is I-equivariant in a suitable sense, so that we can regard its function $\text{Tr}(\text{Fr}_q, \mathcal{G}_1 \star \mathcal{G}_2)$ as an element of the Hecke algebra $\mathcal{H}$.

It is well-known that this product is compatible with the function-sheaf dictionary:

$$[\mathcal{G}_1 \star \mathcal{G}_2] = [\mathcal{G}_1] \star [\mathcal{G}_2].$$

Here $\star$ on the right hand side is just the usual product in $\mathcal{H}$.

Later we shall use the following fact, referred to in the sequel simply as *associativity*: if $\mathcal{G}_i$ $(i = 1, 2, 3)$ are objects of $P_I(\mathcal{F}l)$ such that $\mathcal{G}_1 \star \mathcal{G}_2 \in P_I(\mathcal{F}l)$ and $\mathcal{G}_2 \star \mathcal{G}_3 \in P_I(\mathcal{F}l)$, then there is a canonical isomorphism $\mathcal{G}_1 \star (\mathcal{G}_2 \star \mathcal{G}_3)\tilde{\to}(\mathcal{G}_1 \star \mathcal{G}_2) \star \mathcal{G}_3$ (the "associativity constraint"). This is proved by identifying each canonically with the "triple product" $\mathcal{G}_1 \star \mathcal{G}_2 \star \mathcal{G}_3$, whose construction is similar (see section 6.3).

6.3 Demazure resolution vs. twisted product

It is clear that we can define in a similar way the k-fold convolution product $\star : P_I(\mathcal{F}l)^k \to D^b(\mathcal{F}l)$. To do this we define the k-fold twisted product $X(w_1)\tilde{\times}\cdots\tilde{\times}X(w_k)$ to be the space of k-tuples $(\mathcal{L}_1,\ldots,\mathcal{L}_k)$ such that

$$\mathrm{inv}(\mathcal{L}_{i-1},\mathcal{L}_i) \leq w_i$$

for $1 \leq i \leq k$. If $w = w_1 \cdots w_k$ and $l(w) = \sum_i l(w_i)$, then the multiplication map

$$m : X(w_1)\tilde{\times}\cdots\tilde{\times}X(w_k) \longrightarrow X(w)$$

given by $(\mathcal{L}_1,\ldots,\mathcal{L}_k) \mapsto \mathcal{L}_k$ induces an isomorphism on open subschemes

$$m : Y(w_1)\tilde{\times}\cdots\tilde{\times}Y(w_k)\xrightarrow{\sim}Y(w).$$

If moreover each w_i is a simple reflection s_i, then the twisted product is smooth, since it is a succession of $\mathbb{P}^1$-bundles. We have proved the following lemma.

Lemma 6.1 *If $w = s_1 \cdots s_r$ is a reduced expression, then the twisted product*

$$m : X(s_1)\tilde{\times}\cdots\tilde{\times}X(s_r) \longrightarrow X(w)$$

is a Demazure resolution for the Schubert variety $X(w)$.

We use the k-fold twisted product to define the k-fold convolution product: as before, the objects $\mathcal{G}_i \in P_I(\mathcal{F}l)$ determine a (unique, perverse) twisted exterior product $\mathcal{G}_1\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathcal{G}_k$, and we set

$$\mathcal{G}_1 \star \cdots \star \mathcal{G}_k = m_*(\mathcal{G}_1\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathcal{G}_k).$$

We have the following generalized associativity constraint. Consider a product with k terms

$$(\cdots(\mathcal{G}_1 \star (\cdots \star \mathcal{G}_i))\cdots) \star (\cdots(\mathcal{G}_j \star (\cdots \star \mathcal{G}_k))\cdots)$$

where the placement of the parentheses is arbitrary with the proviso that the product is defined (i.e., at every stage we convolve objects of $P_I(\mathcal{F}l)$). Then this can be identified canonically with the k-fold product

$$\mathcal{G}_1 \star \cdots \star \mathcal{G}_k.$$

This can be seen easily by induction on k.

6.4 Properties of certain convolutions

The convolution of two I-equivariant perverse sheaves on $\mathcal{F}l$ is not perverse in general. However, the following result of I. Mirkovic (unpublished) shows this conclusion does hold in some important cases. We are grateful to R. Bezrukavnikov, who communicated this result to the first author, and to I. Mirkovic, for his kind permission to include the result in this paper.

In the notation of [1], we let $^pD^{\leq 0}(\mathcal{F}l)$ (resp. $^pD^{\geq 0}(\mathcal{F}l)$) denote the objects $P \in D^b(\mathcal{F}l)$ whose perverse cohomology sheaves vanish in degree ≥ 1 (resp. ≤ -1). Thus the perverse sheaves on $\mathcal{F}l$ are precisely the objects in $^pD^{\geq 0}(\mathcal{F}l) \cap {}^pD^{\leq 0}(\mathcal{F}l)$.

Proposition 6.2 (Mirkovic) (a) *Let $P \in P_I(\mathcal{F}l)$. Then for any $w \in \widetilde{W}$, we have*

(1) $J_{w!} \star P$ and $P \star J_{w!}$ belong to $^pD^{\geq 0}(\mathcal{F}l)$,

(2) $J_{w*} \star P$ and $P \star J_{w*}$ belong to $^pD^{\leq 0}(\mathcal{F}l)$.

(b) *In particular, $J_{w_1*} \star J_{w_2!}$ and $J_{w_1!} \star J_{w_2*}$ are perverse, for every $w_1, w_2 \in \widetilde{W}$.*

Proof. (a),(1). We consider $J_{w!} \star P$. Suppose P is supported on $X(w')$ and recall that the convolution is given by

$$J_{w!} \star P = m_!(J_{w!} \tilde{\boxtimes} P),$$

where $m : X(w) \tilde{\times} X(w') \to X$ is as in section 6.2 (note that $m_! = m_*$, since m is proper). We have

$$J_{w!} \star P = m_!(j_{w!}(\mathcal{A}_w) \tilde{\boxtimes} P)$$
$$= [m \circ (j_w \tilde{\times} \mathrm{id})]_!(\mathcal{A}_w \tilde{\boxtimes} P).$$

Note that $m \circ (j_w \tilde{\times} \mathrm{id}) : Y(w) \tilde{\times} X(w') \to X$ is affine, and that $\mathcal{A}_w \tilde{\boxtimes} P$ is perverse on its source. Therefore $J_{w!} \star P \in {}^p D^{\geq 0}(\mathcal{F}l)$ follows from the following general fact (cf. [1], Thm. 4.1.1 and Cor. 4.1.2): If $F : X \to Y$ is an affine morphism, then $F_* : D_c^b(X) \to D_c^b(Y)$ (resp. $F_!$) preserves ${}^p D^{\leq 0}$ (resp. ${}^p D^{\geq 0}$).

A similar argument gives $P \star J_{w!} \in {}^p D^{\geq 0}(\mathcal{F}l)$, once it is noted that $m \circ (\mathrm{id} \tilde{\times} j_w) : X(w') \tilde{\times} Y(w) \to X$ is also an affine morphism.

Part (a),(2) is similar, and part (b) is an immediate consequence of part (a).

∎

Now we let $P_I(\mathcal{F}l) \cap J_{w*} P_I(\mathcal{F}l)$ (resp. $P_I(\mathcal{F}l) \cap J_{w!} P_I(\mathcal{F}l)$) denote the full subcategory of $P_I(\mathcal{F}l)$ whose objects are of the form $J_{w*} \star \mathcal{P}$ (resp. $J_{w!} \star \mathcal{P}$) for some $\mathcal{P} \in P_I(\mathcal{F}l)$.

Corollary 6.3 *For any $w \in \widetilde{W}$, we have*

$$J_{w!} \star J_{w^{-1}*} = J_{w^{-1}*} \star J_{w!} = J_e,$$

where $e \in \widetilde{W}$ is the identity element. Together with the associativity constraint, these identities imply that

$$J_{w!} \star - \; : \; P_I(\mathcal{F}l) \cap J_{w^{-1}*} P_I(\mathcal{F}l) \longrightarrow P_I(\mathcal{F}l) \cap J_{w!} P_I(\mathcal{F}l)$$

is an equivalence of categories, with inverse $J_{w^{-1}*} \star -$.

Proof. We prove that $J_{w!} \star J_{w^{-1}*} = J_e$ (the other equality is similar). Let $X(y)$ be an irreducible component in the support of $P := J_{w!} \star J_{w^{-1}*}$. Since P is perverse and I-equivariant, the restriction $P|Y(y)$ is an I-equivariant ℓ-adic local system on the affine space $Y(y)$. We need to show that $y = e$ and $P|Y(e) = \bar{\mathbb{Q}}_\ell$. For the former it is sufficient to prove that $y \neq e$ implies $P|Y(y) = 0$.

Since $P|Y(y)$ is an I-equivariant ℓ-adic local system on the I-orbit $Y(y)$, and the stabilizer in I of any point in this orbit is geometrically connected, it follows that $P|Y(y)$ is a *constant* local system (placed in degree $-l(y)$ when regarded as a complex). Write $\alpha_1, \alpha_2, \ldots, \alpha_r$ for the eigenvalues of Fr_q on $P|Y(y)[-l(y)]$, counted with multiplicity. We have the following identity for every $n \geq 1$:

$$J_e(y) = \mathrm{Tr}(\mathrm{Fr}_{q^n}, J_{w!}) \star \mathrm{Tr}(\mathrm{Fr}_{q^n}, J_{w^{-1}*})(y) = \mathrm{Tr}(\mathrm{Fr}_{q^n}, P|Y(y)) = \epsilon_y \sum_{i=1}^r \alpha_i^n.$$

We thus have, for every $n \geq 1$:

$$\epsilon_y \sum_{i=1}^r \alpha_i^n = \begin{cases} 0, & \text{if } y \neq e \\ 1, & \text{if } y = e. \end{cases}$$

The linear independence of characters (or rather its proof) implies that distinct numbers $\beta \in \bar{\mathbb{Q}}_\ell^\times$ determine linearly independent characters $n \mapsto \beta^n$ on the semi-group of positive integers. Together with the above formula this is enough information to determine the eigenvalues of Fr_q on $P|(Y(y)$ (if $y \neq e$ then $r = 0$, and if $y = e$ then $r = 1$ and $\alpha_1 = 1$). Thus $P|Y(y) = 0$ if $y \neq e$ and $P|Y(e) = \bar{\mathbb{Q}}_\ell$, as desired.

∎

It is straightforward to check the following properties. From now on we omit the convolution sign $\star$ in the product of perverse sheaves.

Lemma 6.4 *1. If $l(xy) = l(x) + l(y)$, then $J_{x!} J_{y!} = J_{xy!}$.*

2. *Under the same assumption, $J_{x*}J_{y*} = J_{xy*}$.*

Proof. The first can be checked from the definitions, and the second follows on applying Verdier duality. We have used that Verdier duality is compatible with convolution: $D(\mathcal{G}_1 \star \mathcal{G}_2) = D\mathcal{G}_1 \star D\mathcal{G}_2$, if $\mathcal{G}_i \in P_I(\mathcal{F}l)$ $(i = 1, 2)$.

∎

Remark. We remark that Corollary 6.3 and Lemma 6.4 allow us to perform algebraic manipulations involving the perverse sheaf $J_{w!}$ (resp. J_{w*}): essentially it behaves just like its function $\varepsilon_w \tilde{T}_w$ (resp. $\varepsilon_w \tilde{T}_{w^{-1}}^{-1}$) (but when multiplying sheaves, one has to take care that they are each perverse). For example, we have the following cancellation property. Let $P_i \in P_I(\mathcal{F}l)$ $(i = 1, 2)$ be such that $J_{w*} \star P_i \in P_I(\mathcal{F}l)$ $(i = 1, 2)$; then $J_{w*}P_1 \cong J_{w*}P_2$ implies $P_1 \cong P_2$ (multiply both sides by $J_{w^{-1}!}$ and use associativity). We shall use this several times in the proof of Lemma 6.5 below.

6.5 Sheaf analogue Ξ_λ^- of Θ_λ^-

We now define the sheaf-analogue of Θ_λ^-. We write $J_{\lambda*}$ (resp. $J_{\lambda!}$) in place of $J_{t_\lambda*}$ (resp. $J_{t_\lambda!}$). If $\lambda = \lambda_1 - \lambda_2$, where λ_i is anti-dominant $(i = 1, 2)$, then we define

$$\Xi_\lambda^- = J_{\lambda_1!}J_{-\lambda_2*}.$$

By Proposition 6.2, Ξ_λ^- is an object of $P_I(\mathcal{F}l)$. Moreover it clearly satisfies

$$[\Xi_\lambda^-] = \varepsilon_\lambda \Theta_\lambda^-.$$

By [4], we know that $\mathrm{supp}(\Theta_\lambda^-) \subset \{x \,|\, x \leq t_\lambda\}$, and hence that Ξ_λ^- is supported on $X(t_\lambda)$.

Our next goal is to prove the sheaf-theoretic analogues of the relations in the Bernstein presentation of $\mathcal{H}$, in the following lemma. The proof follows Lemma 4.4 of [13] very closely, taking into account Proposition 6.2, Corollary 6.3 and Lemma 6.4. Since a little extra care must be taken in the present context of perverse sheaves, we give detailed arguments for the convenience of the reader.

Lemma 6.5 *(1) If $x, y \in \widetilde{W}$ commute and $l(xy) = l(x) + l(y)$, then $J_{x!}J_{y^{-1}*} = J_{y^{-1}*}J_{x!}$. In particular, if $\mu, \lambda \in X_*$ are both dominant or antidominant, then $J_{\mu!}J_{-\lambda*} = J_{-\lambda*}J_{\mu!}$.*

(2) Ξ_λ^- is independent of the choice of λ_i ($i = 1, 2$).

(3) $\Xi_\lambda^- \Xi_\mu^- = \Xi_{\lambda+\mu}^-$ for $\lambda, \mu \in X_*$.

(4) If $s = s_\alpha$ $(\alpha \in \Pi)$ and $\langle \alpha, \lambda \rangle = 0$, then

$$J_{s*}\Xi_\lambda^- = \Xi_\lambda^- J_{s*}.$$

Moreover, this object belongs to $P_I(\mathcal{F}l)$.

(5) If $\langle \alpha, \lambda \rangle = -1$, then

$$J_{s*}\Xi_\lambda^- J_{s*} = \Xi_{s\lambda}^-.$$

Proof. (1). By Lemma 6.4, we have $J_{x!}J_{y!} = J_{xy!} = J_{yx!} = J_{y!}J_{x!}$. The result follows by two applications of Corollary 6.3: multiply first on the left and then on the right by $J_{y^{-1}*}$.

(2). Let $\lambda = \lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$, where λ_i, λ_i' are antidominant $(i = 1, 2)$. Then $\lambda_2' + \lambda_1 = \lambda_1' + \lambda_2$, and so $J_{\lambda_2'!}J_{\lambda_1!} = J_{\lambda_1'!}J_{\lambda_2!}$. Arguing as in (1), this yields $J_{\lambda_1!}J_{-\lambda_2*} = J_{-\lambda_2'*}J_{\lambda_1'!}$. Then (1) yields $J_{\lambda_1!}J_{-\lambda_2*} = J_{\lambda_1'!}J_{-\lambda_2'*}$.

(3). Write $\lambda = \lambda_1 - \lambda_2$ and $\mu = \mu_1 - \mu_2$, where λ_i, μ_i are antidominant $(i = 1, 2)$. Then by associativity we have

$$
\begin{aligned}
\Xi_\lambda^- \Xi_\mu^- &= (J_{\lambda_1!} J_{-\lambda_2*})(J_{\mu_1!} J_{-\mu_2*}) \\
&= J_{\lambda_1!}(J_{-\lambda_2*} J_{\mu_1!}) J_{-\mu_2*} \\
&= J_{\lambda_1!}(J_{\mu_1!} J_{-\lambda_2*}) J_{-\mu_2*} \\
&= (J_{\lambda_1!} J_{\mu_1!})(J_{-\lambda_2*} J_{-\mu_2*}) \\
&= J_{(\lambda_1+\mu_1)!} J_{-(\lambda_2+\mu_2)*} \\
&= \Xi_{\lambda+\mu}^-.
\end{aligned}
$$

(4). We may write $\lambda = \lambda_1 - \lambda_2$, where λ_i is antidominant and $\langle \alpha, \lambda_i \rangle = 0$, for $i = 1, 2$. Since s commutes with t_{λ_i} and $l(st_{\lambda_i}) = l(t_{\lambda_i}s) = l(t_{\lambda_i}) + 1$, the result follows from (1), Lemma 6.4, and associativity.

We note that $J_{s*}(J_{\lambda_1!} J_{-\lambda_2*}) = J_{st_{-\lambda_2}*} J_{t_{\lambda_1}!}$ is I-equivariant and perverse, by Proposition 6.2.

(5). We may write $\lambda = \lambda_1 - \lambda_2$, where λ_i is antidominant $(i = 1, 2)$, $\langle \alpha, \lambda_1 \rangle = -1$, and $\langle \alpha, \lambda_2 \rangle = 0$.

As in the proof of (4) above, we note that $J_{s*}(J_{\lambda_1!} J_{-\lambda_2*})$ and $(J_{\lambda_1!} J_{-\lambda_2*}) J_{s*}$ are each perverse, so by associativity we may unambiguously write

$$
J_{s*} \Xi_\lambda^- J_{s*} = J_{s*}(J_{\lambda_1!} J_{-\lambda_2*}) J_{s*}.
$$

Using (4) and associativity, this is $(J_{s*} J_{\lambda_1!} J_{s*}) J_{-\lambda_2*} = (J_{s*} \Xi_{\lambda_1}^- J_{s*}) \Xi_{-\lambda_2}^-$. (Note $J_{s*} J_{\lambda_1!} J_{s*}$ is unambiguous and perverse, since $st_{\lambda_1} < t_{\lambda_1}$ implies that $J_{\lambda_1!} J_{s*} = J_{s!} J_{st_{\lambda_1}!} J_{s*}$, and therefore $J_{s*} J_{\lambda_1!} J_{s*} = J_{st_{\lambda_1}!} J_{s*}$.)

Since by (3) $\Xi_{s\lambda_1}^- \Xi_{-\lambda_2}^- = \Xi_{s\lambda}^-$, we are reduced to proving $J_{s*} \Xi_{\lambda_1}^- J_{s*} = \Xi_{s\lambda_1}^-$, i.e., to prove the result for general λ it is enough to consider λ which are antidominant.

Therefore assume λ is antidominant, and write $l(t_\lambda) = l$. Following Lemma 4.4 (b) of [13] we see

- $l(t_\lambda s) = l + 1$ and $l(st_\lambda) = l - 1$,

- $\lambda + s\lambda$ is antidominant,

- $l(t_\lambda st_\lambda) = 2l - 1$ and $l(t_\lambda st_\lambda s) = 2l - 2$; in particular $l(t_\lambda st_\lambda) = l(t_\lambda) + l(st_\lambda)$.

Taking these relations, the previous parts of the Lemma, Corollary 6.3, and associativity into account, we find

$$
\begin{aligned}
J_{\lambda!} \Xi_{s\lambda}^- &= \Xi_\lambda^- \Xi_{s\lambda}^- \\
&= \Xi_{\lambda+s\lambda}^- \\
&= J_{t_\lambda st_\lambda s!} \\
&= J_{t_\lambda st_\lambda !} J_{s*} \\
&= J_{\lambda!} J_{st_\lambda !} J_{s*} \\
&= J_{\lambda!}(J_{s*} J_{\lambda!} J_{s*}).
\end{aligned}
$$

Using Corollary 6.3 again to cancel $J_{\lambda!}$ from each side, we obtain the desired equality $J_{s*} \Xi_\lambda^- J_{s*} = \Xi_{s\lambda}^-$.

∎

Note that property (5) is the analogue of Bernstein's relation

$$
\tilde{T}_s^{-1} \Theta_\lambda^- \tilde{T}_s^{-1} = \Theta_{s\lambda}^-,
$$

which was a main ingredient in the proof of Lemma 3.2. In fact the same argument can be applied to prove the following corollary.

Corollary 6.6 *If λ is minuscule and $t_\lambda = w^\lambda w$ as in section 3, then*

$$
\Xi_\lambda^- = J_{w^\lambda!} J_{w*}.
$$

Writing $w^\lambda = t_1 \cdots t_{r-p} \tau$ and $w = s_1 \cdots s_p$ as in section 3, we have

$$
\Xi_\lambda^- = J_{t_1!} \cdots J_{t_{r-p}!} J_{\tau!} J_{s_1*} \cdots J_{s_p*}.
$$

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Proof. Suppose λ is in the Weyl orbit of an antidominant minuscule coweight μ^-. We have

$$\Xi_{\mu^-}^- = J_{\mu^-!}.$$

Choose the sequence of simple reflections $s_1, \ldots, s_p$ as in Lemma 3.2. By induction on p, we easily see that

$$\Xi_{s_p \cdots s_1(\mu^-)}^- = J_{t_1 \cdots t_{r-p} \tau!} J_{s_1 \cdots s_p *}.$$

Indeed, using induction and Lemma 6.4 this equality for $p-1$ can be written

$$\Xi_{s_{p-1} \cdots s_1(\mu^-)}^- = J_{s_p!} J_{t_1 \cdots t_{r-p} \tau!} J_{s_1 \cdots s_{p-1} *}.$$

Multiplying on each side by $J_{s_p *}$ and using Lemma 6.5 (5), Corollary 6.3, and associativity yields

$$\Xi_{s_p \cdots s_1(\mu^-)}^- = J_{t_1 \cdots t_{r-p} \tau!} J_{s_1 \cdots s_p *},$$

as desired.

The second statement follows from the first, using Lemma 6.4. To justify this, we need to show that

$$(J_{t_1!} \cdots J_{t_{r-p}!} J_{\tau!})(J_{s_1 *} \cdots J_{s_p *}) = J_{t_1!} \cdots J_{t_{r-p}!} J_{\tau!} J_{s_1 *} \cdots J_{s_p *},$$

where the products of the form $J_{t_1!} \cdots J_{s_p *}$ denote the k-fold convolution mentioned in section 6.3. This results from the generalized associativity discussed there.

We remark that it is important here that the underlying expression $t_1 \cdots t_{r-p} \tau s_1 \cdots s_p$ is reduced. ∎

Taking Corollary 6.6 as well as Lemmas 6.1, 6.4 and 6.5 into account, we get the following sheaf-theoretic interpretation for minimal expressions for Θ_λ^-.

Theorem 6.7 (a) *Let λ be a minuscule coweight of any root system. Write $t_\lambda = w^\lambda w = t_1 \cdots t_{r-p} \tau s_1 \cdots s_p$, as in Lemma 3.2. Then*

$$\Xi_\lambda^- = m_*(\mathcal{D}),$$

where $\mathcal{D}$ is the perverse sheaf

$$\mathcal{D} = (j_{t_1!} \mathcal{A}_{t_1}) \tilde{\boxtimes} \cdots \tilde{\boxtimes} (j_{t_{r-p}!} \mathcal{A}_{t_{r-p}}) \tilde{\boxtimes} (j_{\tau!} \mathcal{A}_\tau) \tilde{\boxtimes} (j_{s_1 *} \mathcal{A}_{s_1}) \tilde{\boxtimes} \cdots \tilde{\boxtimes} (j_{s_p *} \mathcal{A}_{s_p})$$

on the Demazure resolution $m : X(t_1) \tilde{\times} \cdots \tilde{\times} X(t_{r-p}) \tilde{\times} X(\tau) \tilde{\times} X(s_1) \tilde{\times} \cdots \tilde{\times} X(s_p) \to X(t_\lambda)$ of the Schubert variety $X(t_\lambda)$.
(b) Let λ be a coweight for GL_n, and write it as

$$\lambda = \lambda_1 + \cdots + \lambda_k,$$

where each λ_i is minuscule, and $l(t_\lambda) = \sum_i l(t_{\lambda_i})$. For each $i = 1, \ldots, k$ we have a decomposition and reduced expression

$$t_{\lambda_i} = w^{\lambda_i} w_i = t_1^i \cdots t_{q_i}^i \tau^i s_1^i \cdots s_{p_i}^i,$$

as in Lemma 3.2.
Then $\Xi_\lambda^- = m_*(\mathcal{D})$, where $\mathcal{D}$ is the perverse sheaf

$$J_{t_1^1!} \tilde{\boxtimes} \cdots \tilde{\boxtimes} J_{t_{q_1}^1!} \tilde{\boxtimes} J_{\tau^1!} \tilde{\boxtimes} J_{s_1^1 *} \tilde{\boxtimes} \cdots \tilde{\boxtimes} J_{s_{p_1}^1 *} \tilde{\boxtimes} \cdots \cdots \cdots \tilde{\boxtimes} J_{t_1^k!} \tilde{\boxtimes} \cdots \tilde{\boxtimes} J_{t_{q_k}^k!} \tilde{\boxtimes} J_{\tau^k!} \tilde{\boxtimes} J_{s_1^k *} \tilde{\boxtimes} \cdots \tilde{\boxtimes} J_{s_{p_k}^k *}$$

on the Demazure resolution $m : \tilde{X}(t_\lambda) \to X(t_\lambda)$ corresponding to the reduced expression

$$t_\lambda = (t_1^1 \cdots t_{q_1}^1 \tau^1 s_1^1 \cdots s_{p_1}^1) \cdots (t_1^k \cdots t_{q_k}^k \tau^k s_1^k \cdots s_{p_k}^k).$$

Consequently, Ξ_λ^- *is the push-forward of an explicit perverse sheaf on a Demazure resolution of $X(t_\lambda)$, for every coweight λ of GL_n.*

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Proof. (a). This follows directly from Corollary 6.6 and the definition of the r-fold convolution product. (b). By Lemma 6.5 (3) and generalized associativity, we have

$$\Xi_\lambda^- = \Xi_{\lambda_1}^- \star \cdots \star \Xi_{\lambda_k}^-,$$

where the right hand side denotes the k-fold convolution product.

Part (b) then follows from part (a) and another application of generalized associativity.
∎

For $x \leq t_\lambda$, write $\Theta_\lambda^-(x)$ for the coefficient of T_x.

Corollary 6.8 *Let λ be a minuscule coweight of a root system, or an arbitrary coweight for GL_n. Let $m, \mathcal{D}$ be as in Theorem 6.7. Then*

$$\Theta_\lambda^-(x) = \varepsilon_\lambda \mathrm{Tr}(\mathrm{Fr}_q, H^\bullet(m^{-1}(x), \mathcal{D})),$$

where the right hand side is the alternating trace of Frobenius on the étale cohomology of the fiber over x with coefficients in $\mathcal{D}$.

7 Acknowledgements

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