STRUCTURE CONSTANTS FOR HECKE
AND REPRESENTATION RINGS

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ABSTRACT. In [KLM] the authors study certain structure constants for two related
rings: the spherical Hecke algebra of a split connected reductive group over a local
non-Archimedean field, and the representation ring of the Langlands dual group. The
former are defined relative to characteristic functions of double cosets, and the latter
relative to highest weight representations. They prove that the nonvanishing of one
of the latter structure constants always implies the nonvanishing of the corresponding
former one. For GL_n, the reverse implication also holds, and is due to P. Hall. Both
proofs are combinatorial in nature. In this note, we provide geometric proofs of
both results, using affine Grassmannians. We also provide some additional results
concerning minuscule coweights and the equidimensionality of the fibers of certain
Bott-Samelson resolutions of affine Schubert varieties for GL_n.

1. INTRODUCTION

Given a split connected reductive group G over a local non-Archimedean field F,
one may consider two related rings: the spherical Hecke algebra attached to $G(F)$, and
the representation ring of the Langlands dual group $\hat{G}$. By the Satake isomorphism,
these rings are actually isomorphic. However, the two realizations give rise naturally
to two different bases and thus to two different sets of structure constants defining this
ring. For the Hecke algebra realization, consider the characteristic functions of double
cosets with respect to a special maximal compact subgroup; for the representation
ring realization, consider the highest weight representations. In [KLM], the authors
prove several new facts concerning these structure constants and their relation to each
other, culminating in a new proof of the saturation property for the group GL_n. They
prove that the non-vanishing of a representation ring structure constant implies the
non-vanishing of the corresponding Hecke algebra structure constant. For GL_n, the
converse implication also holds, and is due to P. Hall. Both proofs are combinatiorial
in nature.

The purpose of this paper is to provide geometric proofs for both results, using
affine Grassmannians. Along the way, we provide some additional results concerning
minuscule coweights and the equidimensionality of the fibers of certain Bott-Samelson
resolutions of affine Schubert varieties for GL_n. These geometric methods avoid the
combinatorial tools which are specific to the group GL_n, and so give some indications
how one may approach proving saturation properties for more general groups.

More precisely, let G denote a connected reductive group over an algebraic closure k
of a finite field $\mathbb{F}_q$. We assume G is defined and split over $\mathbb{F}_q$. Let T denote an $\mathbb{F}_q$-split

maximal torus of G. Let $B = TU$ denote an $\mathbb{F}_q$-rational Borel subgroup containing T. Let $X_*(T)$ denote the set of cocharacters of T, and let $X_+^\vee$ denote the set of B-dominant elements of $X_*(T)$. Let ρ denote half the sum of the B-positive roots for G. Let $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ denote the canonical pairing.

Fix a prime ℓ different from the characteristic of k, and let $\hat{G} = \hat{G}(\bar{\mathbb{Q}}_\ell)$ denote the Langlands dual of G over the field $\bar{\mathbb{Q}}_\ell$ (an algebraic closure of $\mathbb{Q}_\ell$). Each dominant cocharacter $\mu \in X_+^\vee$ can be thought of as a dominant weight for $\hat{G}$, hence such a cocharacter gives rise to a unique irreducible $\bar{\mathbb{Q}}_\ell$-linear representation V_μ of $\hat{G}$ having highest weight μ. Let $\text{Rep}(\hat{G})$ denote the category of rational representations of $\hat{G}$ over $\bar{\mathbb{Q}}_\ell$.

We will work with the spherical Hecke algebra of G in the function-field setting. Let $\mathcal{O} = k[[t]]$ and $F = k((t))$. Also, let $\mathcal{O}_q = \mathbb{F}_q[[t]]$ and $F_q = \mathbb{F}_q((t))$. Let $K := G(\mathcal{O})$ denote the "maximal compact" subgroup of $G(F)$. Let $K_q = G(\mathcal{O}_q)$. We let $\mathcal{H}_q$ denote the Hecke algebra for G over F_q, i.e., the convolution algebra of compactly-supported K_q-bi-invariant $\bar{\mathbb{Q}}_\ell$-valued functions on $G(F_q)$ (convolution is defined using the Haar measure on $G(F_q)$ which assigns measure one to the compact open subgroup K_q). For $\mu \in X_+^\vee$, define an element $f_\mu \in \mathcal{H}_q$ by

$$f_\mu = \text{char}(K_q \, t_\mu \, K_q)$$

where $t_\mu := \mu(t) \in T(F_q)$. By the Cartan decomposition, the elements f_μ form a $\bar{\mathbb{Q}}_\ell$-basis for $\mathcal{H}_q$. For later use let us recall that the convolution operation $*$ is defined, for $f_1, f_2 \in \mathcal{H}_q$, by

$$(f_1 * f_2)(g) = \int_{G(F_q)} f_1(x) f_2(x^{-1}g) dx.$$

Now let $\mu_1, \ldots, \mu_r$ and λ be dominant coweights for G. Denote $\mu_\bullet = (\mu_1, \ldots, \mu_r)$ and $|\mu_\bullet| = \mu_1 + \cdots + \mu_r$. We consider the decomposition in $\text{Rep}(\hat{G})$

$$V_{\mu_1} \otimes \cdots \otimes V_{\mu_r} = \bigoplus_{\lambda \in X_+^\vee, \, \lambda \leq |\mu_\bullet|} V_\lambda \otimes V_{\mu_\bullet}^\lambda,$$

where $V_{\mu_\bullet}^\lambda$ denotes the multiplicity vector space. Here $\lambda_1 \leq \lambda_2$ means that $\lambda_2 - \lambda_1$ is a sum of B-positive coroots. (It follows from the discussion in section 2 that if V_λ occurs as a summand, then $\lambda \leq |\mu_\bullet|$ necessarily holds.) The numbers $\dim(V_{\mu_\bullet}^\lambda)$ are the *structure constants for the representation ring of* $\hat{G}$.

In a parallel manner, for $\mu_\bullet$ and λ as above, we write

$$f_{\mu_1} * \cdots * f_{\mu_r} = \sum_\lambda c_{\mu_\bullet}^\lambda f_\lambda,$$

for certain constants $c_{\mu_\bullet}^\lambda$; in this paper these are referred to as the *structure constants for the Hecke algebra* $\mathcal{H}_q$. It is known and easy to see that the structure constants all belong to $\mathbb{Z}$. In fact, as q varies, they are given by polynomial functions in q with integer coefficients: $c_{\mu_\bullet}^\lambda = c_{\mu_\bullet}^\lambda(q)$. (See Lemmas 9.15, 9.18 of [KLM]).

The purpose of this paper is to study the relation between the following two properties of a collection $(\mu_\bullet, \lambda)$ of dominant cocharacters:

Rep$(\mu_\bullet, \lambda)$: The irreducible representation V_λ occurs with non-zero multiplicity in $V_{\mu_1} \otimes \cdots \otimes V_{\mu_r}$. That is, $\dim(V_{\mu_\bullet}^\lambda) > 0$.

Hecke$(\mu_\bullet, \lambda)$: The function f_λ appears with non-zero coefficient in $f_{\mu_1} * \cdots * f_{\mu_r}$. That is, $c_{\mu_\bullet}^\lambda \neq 0$. Equivalently, $K_q\, t_\lambda\, K_q \subset K_q\, t_{\mu_1}\, K_q \cdots K_q\, t_{\mu_r}\, K_q$.

The following result was first proved for general groups G by M. Kapovich, B. Leeb, and J. Millson in [KLM], Theorem 9.19. In this paper we give another approach, based on the geometry of affine Grassmannians.

Theorem 1.1 (KLM). *If* **Rep**$(\mu_\bullet, \lambda)$ *holds, then* **Hecke**$(\mu_\bullet, \lambda)$ *also holds. More precisely, we have the following description of the Hecke algebra structure constants*

$$c_{\mu_\bullet}^\lambda(q) = \dim(V_{\mu_\bullet}^\lambda)\, q^{\langle \rho, |\mu_\bullet| - \lambda \rangle} + \{terms\ with\ lower\ q\text{-}degree\}.$$

For the general linear group, the converse implication also holds, and is originally due to P. Hall, using the combinatorics of Hall polynomials (cf. [KLM], section 9.6, and [Mac], part II, Theorem (4.3)). We present a geometric proof here, again using affine Grassmannians.

Theorem 1.2 (P. Hall). *Let* $G = \mathrm{GL}_n$. *Then* **Hecke**$(\mu_\bullet, \lambda) \Rightarrow$ **Rep**$(\mu_\bullet, \lambda)$.

The approach of this paper is to reformulate the two properties **Hecke**$(\mu_\bullet, \lambda)$ and **Rep**$(\mu_\bullet, \lambda)$ in terms of properties of the "multiplication" morphism

$$m_{\mu_\bullet} : \widetilde{\mathcal{Q}}_{\mu_\bullet} \to \bar{\mathcal{Q}}_{|\mu_\bullet|}$$

used to define the convolution of K-equivariant perverse sheaves on the affine Grassmannian. Here the domain is a twisted product of the closures $\bar{\mathcal{Q}}_{\mu_i}$ of K-orbits $\mathcal{Q}_{\mu_i}$ in the affine Grassmannian for G. The morphism $m_{\mu_\bullet}$ is given by forgetting all but the last factor in the twisted product. It is a (stratified) semi-small, proper, and birational morphism (cf. section 2). One can view it as a partial desingularization of $\bar{\mathcal{Q}}_{|\mu_\bullet|}$, directly analogous to the Bott-Samelson partial desingularizations of Schubert varieties. (When the coweights μ_i are all minuscule, the domain is smooth, so in that case $m_{\mu_\bullet}$ is a genuine resolution of singularities.) The semi-smallness of $m_{\mu_\bullet}$ means that the fiber over any point y in a K-orbit stratum $\mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$ has dimension bounded above by $\langle \rho, |\mu_\bullet| - \lambda \rangle$.

The two properties can be translated into properties of the fibers of $m_{\mu_\bullet}$. The geometric Satake isomorphism (cf. [Gi], [MV], [NP]) plays a key role in this step.

Proposition 1.3. *Suppose* $(\mu_\bullet, \lambda)$ *is such that* $\mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$ *(equivalently,* $\lambda \leq |\mu_\bullet|$*). Fix* $y \in \mathcal{Q}_\lambda$. *Then:*

1. $V_{\mu_\bullet}^\lambda \neq 0$ *if and only if* $m_{\mu_\bullet}^{-1}(y)$ *has an irreducible component of dimension* $\langle \rho, |\mu_\bullet| - \lambda \rangle$. *Moreover, such a component necessarily meets the open stratum* $\mathcal{Q}_{\mu_\bullet}$.

2. Suppose y *is* $\mathbb{F}_q$-*rational. Then* f_λ *occurs in* $f_{\mu_1} * \cdots * f_{\mu_n}$ *if and only if* $m_{\mu_\bullet}^{-1}(y)$ *meets the open stratum* $\mathcal{Q}_{\mu_\bullet}$. *In that case, we have*

$$c_{\mu_\bullet}^\lambda(q) = \#(m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu_\bullet})(\mathbb{F}_q) > 0.$$

The first part of Theorem 1.1 follows immediately from Proposition 1.3. We deduce the second part concerning $c_{\mu_\bullet}^\lambda$ with little additional effort, by using the Weil conjectures to approximate the number of points on the algebraic varieties $m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu_\bullet}$.

We also have the following partial converse to Theorem 1.1, valid for every group G. It is an easy and purely combinatorial consequence of the P-R-V conjecture (now a theorem due independently to S. Kumar and O. Mathieu, cf. [Lit]).

Proposition 1.4. *If* $\mathbf{Hecke}(\mu_\bullet, \lambda)$ *holds, then there exist dominant coweights* $\mu_i' \leq \mu_i$ *($1 \leq i \leq r$), such that* $\mathbf{Rep}(\mu_\bullet', \lambda)$ *holds.*

To prove Theorem 1.2, we need finer information about the fibers of $m_{\mu_\bullet}$ in the GL_n case. The first ingredient is the following proposition valid for all groups having minuscule coweights (recall that a coweight μ is *minuscule* provided that $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$, for every root α of G). It is also a consequence of the P-R-V conjecture.

Proposition 1.5. *If* $\mu_1, \ldots, \mu_r$ *are all minuscule coweights for* G, *and* $\lambda \leq |\mu_\bullet|$, *then both* $\mathbf{Rep}(\mu_\bullet, \lambda)$ *and* $\mathbf{Hecke}(\mu_\bullet, \lambda)$ *hold.*

In view of Proposition 1.3, this implies:

Corollary 1.6. *Let* $\mu_1, \ldots, \mu_r$ *let be minuscule coweights for* G, *and let* $\lambda \leq |\mu_\bullet|$. *Then for any* $y \in \mathcal{Q}_\lambda$, *we have*

$$\dim(m_{\mu_\bullet}^{-1}(y)) = \langle \rho, |\mu_\bullet| - \lambda \rangle.$$

In particular, every stratum of $\bar{\mathcal{Q}}_{|\mu_\bullet|}$ *is relevant for the semi-small morphism* $m_{\mu_\bullet}$.

We make essential use of the further information that in the GL_n case, the fibers above are *equidimensional*.

Proposition 1.7. *Let* $G = \mathrm{GL}_n$. *Let* $\mu_1, \ldots, \mu_r$ *be minuscule coweights. Suppose* $\mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$, *and let* $y \in \mathcal{Q}_\lambda$. *Then every irreducible component of the fiber*

$$m_{\mu_\bullet}^{-1}(y)$$

has dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.

The proof of Proposition 1.7 proceeds by reduction to a theorem of N. Spaltenstein [Sp] concerning partial Springer resolutions of the nilpotent cone for GL_n. The lack of an analogous result for other groups is one reason Proposition 1.7 can be proved (at the moment) only for GL_n.

In fact for the general linear group, Proposition 1.7 can be used to prove a seemingly stronger result.

Proposition 1.8. *Let* $\mu_1, \ldots, \mu_r$ *be dominant coweights for* $G = \mathrm{GL}_n$. *Let* $y \in \mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$. *Let* $\mathcal{Q}_{\mu_\bullet'} \subset \widetilde{\mathcal{Q}}_{\mu_\bullet}$ *be the stratum indexed by* $\mu_\bullet' = (\mu_1', \ldots, \mu_r')$ *for dominant coweights* $\mu_i' \leq \mu_i$ *($1 \leq i \leq r$). Then any irreducible component of the fiber* $m_{\mu_\bullet}^{-1}(y)$ *whose generic point belongs to* $\mathcal{Q}_{\mu_\bullet'}$ *has dimension* $\langle \rho, |\mu_\bullet'| - \lambda \rangle$.

As we explain in section 8, Proposition 1.8 quickly implies Theorem 1.2.

Proposition 1.8 appears to be special to GL_n. Indeed, it is actually somewhat stronger than the implication $\mathbf{Hecke}(\mu_\bullet, \lambda) \Rightarrow \mathbf{Rep}(\mu_\bullet, \lambda)$, which is known to fail in general (e.g. for $SO(5)$ or G_2, cf. [KLM], section 9.5). But Proposition 1.7 could remain valid if GL_n is replaced with an arbitrary group G, and it would be interesting to clarify the situation. Such a generalization of Proposition 1.7 would have applications to proving a type of "Saturation theorem" for a general reductive group (along the same lines as the new proof in [KLM] of the Saturation theorem for GL_n, originally proved by Knutson-Tao [KT]).

Remark. Suitably reformulated, the main results of this paper hold if the coefficient field $k = \bar{\mathbb{F}}_q$ is replaced by any algebraically closed field κ (e.g. $\mathbb{C}$). We let now $\mathcal{O} = \kappa[[t]]$, $K = G(\mathcal{O})$, etc., and replace $\mathbf{Hecke}(\mu_\bullet, \lambda)$ with

$\mathbf{Hecke}'(\mu_\bullet, \lambda) : Kt_\lambda K \subset Kt_{\mu_1} K \cdots Kt_{\mu_r} K.$

Then the arguments of this paper prove that $\mathbf{Rep}(\mu_\bullet, \lambda) \Rightarrow \mathbf{Hecke}'(\mu_\bullet, \lambda)$ for every G, and that $\mathbf{Rep}(\mu_\bullet, \lambda) \Leftrightarrow \mathbf{Hecke}'(\mu_\bullet, \lambda)$ for GL_n. The statements concerning dimensions of fibers and their equidimensionality also remain valid. Thus, here we avoid the hypothesis that F_q have finite residue field, essential for Hall's proof of Theorem 1.2 using Hall polynomials, and for the proof of Theorem 1.1 in ([KLM], Theorem 9.19).

2. Review of affine Grassmannians

In this section we recall some well-known notions relating to affine Grassmannians. The reader can find further details in [Gi], [MV], and [NP].

We will work with the affine Grassmannian

$$\mathcal{Q} = G(F)/K,$$

which can be thought of as the k-points of an ind-scheme defined over $\mathbb{F}_q$. The group scheme $K = G(\mathcal{O})$ acts naturally on $\mathcal{Q}$ (on the left). By the Cartan decomposition, the K-orbits are parametrized by dominant coweights $\lambda \in X_+^\vee$. Indeed, let e_0 denote the base point of $\mathcal{Q}$ corresponding to the coset K, and let $e_\lambda = t_\lambda e_0$. Then $\mathcal{Q}_\lambda = Ke_\lambda$ is the K-orbit corresponding to λ. It is well-known that $\mathcal{Q}_\lambda$ is a smooth quasi-projective variety of dimension $\langle 2\rho, \lambda \rangle$, defined over $\mathbb{F}_q$. Its closure $\bar{\mathcal{Q}}_\lambda$ in $\mathcal{Q}$ is projective, but in general is not smooth.

Let $j_\lambda : \mathcal{Q}_\lambda \hookrightarrow \bar{\mathcal{Q}}_\lambda$ denote the open immersion into the closure, and let us define the intersection complex

$$\mathcal{A}_\lambda := \mathrm{IC}_\lambda(\bar{\mathbb{Q}}_\ell) = j_{\lambda,!*}(\bar{\mathbb{Q}}_\ell)[\langle 2\rho, \lambda \rangle].$$

Here we are applying $j_{\lambda,!*}$, the Goresky-MacPherson middle extension functor (for the middle perversity, cf. [GM], [BBD]), to the shifted constant sheaf on the smooth variety $\mathcal{Q}_\lambda$. The cohomological shift by the dimension of $\mathcal{Q}_\lambda$ is to ensure the result is a perverse sheaf. It is known that $\mathcal{A}_\lambda$ is a self-dual K-equivariant simple perverse sheaf on $\mathcal{Q}$.

Let $P_K(\mathcal{Q})$ denote the category of $\bar{\mathbb{Q}}_\ell$-linear K-equivariant perverse sheaves on $\mathcal{Q}$. This is a semi-simple abelian category (cf. [Ga]), whose simple objects are precisely the intersection complexes $\mathcal{A}_\lambda$, for $\lambda \in X_+^\vee$. In fact there exists a tensor (or "convolution") operation

$$\star : P_K(\mathcal{Q}) \times P_K(\mathcal{Q}) \to P_K(\mathcal{Q})$$

(defined below) which gives $P_K(\mathcal{Q})$ the structure of a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. The following theorem which identifies the corresponding algebraic group as the Langlands dual plays a key role in this paper. We refer the reader to [Gi],[MV], [NP], and the appendix of [Nad], for details of the proof.

Theorem 2.1 (Geometric Satake Isomorphism). *There is an equivalence of tensor categories*

$$(P_K(\mathcal{Q}), \star) \xrightarrow{\;\sim\;} (\mathrm{Rep}(\hat{G}), \otimes),$$

under which $\mathcal{A}_\lambda$ corresponds to the irreducible representation V_λ with highest weight λ.

2.1. **Definition of the twisted product.** To define the operation $\star$, we need some more preliminaries. First, recall that any ordered pair of elements $L, L' \in \mathcal{Q}$ gives rise to an element $\mathrm{inv}(L, L') \in X_+^\vee$ (the "relative position" of L, L'). The map $\mathrm{inv} : \mathcal{Q} \times \mathcal{Q} \to X_+^\vee$ is defined as the composition

$$\mathrm{inv} : G(F)/K \times G(F)/K \to G(F) \backslash \big[G(F)/K \times G(F)/K \big] = K \backslash G(F)/K \xrightarrow{\sim} X_+^\vee,$$

where $G(F)$ acts diagonally on $G(F)/K \times G(F)/K$. For $G = \mathrm{GL}_n$, $\mathrm{inv}(L, L')$ is just the usual relative position of two $\mathcal{O}$-lattices in F^n given by the theory of elementary divisors.

The usual partial ordering $\leq$ on the set of dominant coweights corresponds to the closure relation in $\mathcal{Q}$: $\mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_\mu$ if and only if $\lambda \leq \mu$. It follows that $\mathrm{inv}(L, L') \leq \mu$ if and only if there exists $g \in G(F)$ such that $gL = e_0$ and $gL' \in \bar{\mathcal{Q}}_\mu$. Thus, for $L \in \mathcal{Q}$ and $\mu \in X_+^\vee$ fixed, the set of L' with $\mathrm{inv}(L, L') \leq \mu$ can be thought of as the k-points of a projective algebraic variety, isomorphic to $\bar{\mathcal{Q}}_\mu$.

Now let $\mu_\bullet = (\mu_1, \dots, \mu_r)$, where $\mu_i \in X_+^\vee$ for $1 \leq i \leq r$. We define the twisted product scheme

$$\widetilde{\mathcal{Q}}_{\mu_\bullet} = \bar{\mathcal{Q}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \bar{\mathcal{Q}}_{\mu_n}$$

to be the subscheme of $\mathcal{Q}^n$ consisting of points $(L_1, \dots, L_r)$ such that $\mathrm{inv}(L_{i-1}, L_i) \leq \mu_i$ for $1 \leq i \leq r$ (letting $L_0 = e_0$). Projection onto the last coordinate gives a proper surjective birational map

$$m_{\mu_\bullet} : \widetilde{\mathcal{Q}}_{\mu_\bullet} \to \bar{\mathcal{Q}}_{|\mu_\bullet|}.$$

The birationality follows from the more precise statement that the restriction of $m_{\mu_\bullet}$ to the inverse image of the open stratum $\mathcal{Q}_{|\mu_\bullet|} \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$ is an isomorphism. This in turn can be deduced from the corresponding statement for an analogous *Demazure resolution* of an affine Schubert variety in the affine flag variety. We omit the details.

2.2. Semi-small morphisms.

We shall make use of the notion of semi-small morphisms. Let $f : X = \cup_\alpha X_\alpha \to Y = \cup_\beta Y_\beta$ be a proper surjective birational morphism between stratified spaces. Suppose each $f(X_\alpha)$ is a union of strata Y_β. We say f is *semi-small* if, whenever $y \in Y_\beta \subset f(X_\alpha)$, then

$$\dim(f^{-1}(y) \cap X_\alpha) \leq \frac{1}{2}(\dim(X_\alpha) - \dim(Y_\beta)).$$

(In [MV], this notion is termed "stratified semi-small". In the usual terminology ([GM]), the domain of a semi-small morphism is assumed to be smooth.)

We say f is *locally trivial* (in the stratified sense) if whenever $Y_\beta \subset f(X_\alpha)$, the restriction of $f : f^{-1}(Y_\beta) \to Y_\beta$ to $X_\alpha \cap f^{-1}(Y_\beta)$ is Zariski-locally a trivial fibration. In this case we have, for every $y \in Y_\beta \subset f(X_\alpha)$:

$$\dim(f^{-1}(y) \cap X_\alpha) + \dim(Y_\beta) = \dim(f^{-1}(Y_\beta) \cap X_\alpha).$$

The target $\bar{\mathcal{Q}}_{|\mu_\bullet|}$ of $m_{\mu_\bullet}$ is stratified by the locally-closed subschemes $\mathcal{Q}_\lambda$ ($\lambda \leq |\mu_\bullet|$), and the domain $\bar{\mathcal{Q}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \bar{\mathcal{Q}}_{\mu_n}$ is stratified by the locally-closed subspaces $\mathcal{Q}_{\mu'_\bullet} = \mathcal{Q}_{\mu'_1} \tilde{\times} \cdots \tilde{\times} \mathcal{Q}_{\mu'_n}$, where $\mu'_\bullet = (\mu'_1, \ldots, \mu'_n)$ satisfies $\mu'_i \leq \mu_i$, for every i. (The definition of the subspace is the same as that of the ambient space, except that the inequalities $\mathrm{inv}(L_{i-1}, L_i) \leq \mu_i$ are replaced by the equalities $\mathrm{inv}(L_{i-1}, L_i) = \mu'_i$.) With respect to these stratifications, we have the following result.

Proposition 2.2 ([MV], [NP]). *The morphism $m_{\mu_\bullet}$ is a semi-small and locally trivial morphism.*

Remark. The proof of semi-smallness over $\mathbb{F}_q$ does not appear explicitly in the literature in precisely this generality. The proof of Lemme 9.3 in [NP] for the special case where all μ_i are minuscule or quasi-minuscule works as well for the general case, modulo the inequality $\dim(S_\nu \cap \mathcal{Q}_\lambda) \leq \langle \rho, \lambda + \nu \rangle$, where $\nu \in X_*(T)$ and $S_\nu := U(F)e_\lambda$. In fact the equality $\dim(S_\nu \cap \mathcal{Q}_\lambda) = \langle \rho, \lambda + \nu \rangle$ can be deduced (as Ngô and Polo remark), from their Theoreme 3.1. This indirect method to prove the dimension equality is not circular, since Ngô and Polo use the equality $\dim(S_\nu \cap \mathcal{Q}_\lambda) = \langle \rho, \lambda + \nu \rangle$ only in the special case mentioned above in their proof of Theoreme 3.1, and they prove this special case by direct means.

2.3. Convolution of perverse sheaves.

If $\mathcal{G}_i$ ($1 \leq i \leq r$) are elements of $P_K(\mathcal{Q})$, choose coweights μ_i such that $\mathrm{supp}(\mathcal{G}_i) \subset \bar{\mathcal{Q}}_{\mu_i}$, for every i (technical aside: this is possible, as we can assume with no loss of generality that each $\mathcal{G}_i$ is supported on only one connected component of $\mathcal{Q}$). There is a unique "twisted product" perverse sheaf $\mathcal{G}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{G}_r$ on the twisted product $\tilde{\mathcal{Q}}_{\mu_\bullet}$, which is locally isomorphic to the usual

exterior product $\mathcal{G}_1 \boxtimes \cdots \boxtimes \mathcal{G}_r$ on the product space $\bar{\mathcal{Q}}_{\mu_1} \times \cdots \times \bar{\mathcal{Q}}_{\mu_r}$. (See 7.4 of [HN], or [NP], section 2, for another construction). We then define

$$\mathcal{G}_1 \star \cdots \star \mathcal{G}_r = Rm_{\mu_\bullet, *}(\mathcal{G}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{G}_r).$$

This belongs to $P_K(\mathcal{Q})$ by the semi-smallness of $m_{\mu_\bullet}$, and is independent of the choice of $\mu_\bullet$.

The important lemma below follows directly from the definitions and the following characterization of the intersection complex ([BBD], 2.1.17): Let X^0 denote the open (smooth) stratum in a stratified variety $X = \cup_\alpha X_\alpha$. Then $\mathrm{IC}(X)$ is the unique self-dual perverse extension of $\bar{\mathbb{Q}}_\ell[\dim(X)]$ on X^0 which satisfies, for each stratum $X_\alpha \neq X^0$, the property $\mathcal{H}^i \mathrm{IC}(X)|_{X_\alpha} = 0$ for $i \geq -\dim(X_\alpha)$.

Lemma 2.3. $\mathcal{A}_{\mu_1} \star \cdots \star \mathcal{A}_{\mu_r} = Rm_{\mu_\bullet, *} \mathrm{IC}(\widetilde{\mathcal{Q}}_{\mu_\bullet})$.

We are going to use this to give a geometric description of the multiplicity space $V^\lambda_{\mu_\bullet}$, which will be the key step to proving Proposition 1.3.

3. Proof of Proposition 1.3, Part (1)

The key result is the following well-known proposition. It has appeared without proof in several published articles (e.g. [NP]). We provide a proof here for the convenience of the reader.

Proposition 3.1. *Fix* $\lambda \leq |\mu_\bullet|$, *and* $y \in \mathcal{Q}_\lambda$. *Then the space* $V^\lambda_{\mu_\bullet}$ *has a basis in canonical bijection with the set of irreducible components of* $m^{-1}_{\mu_\bullet}(y)$ *having maximal possible dimension, that is,* $\langle \rho, |\mu_\bullet| - \lambda \rangle$.

Let us assume Proposition 3.1 and deduce Proposition 1.3, Part (1). The first statement is obvious, so we verify the second. But it is clear that an irreducible component C of $m^{-1}_{\mu_\bullet}(y)$ with dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$ meets the open stratum $\mathcal{Q}_{\mu_\bullet}$ (if not, then C meets only strictly smaller strata $\mathcal{Q}_{\mu'_\bullet}$, and then by Proposition 2.2, the dimension of C would be bounded above by

$$\dim(m^{-1}_{\mu_\bullet}(y) \cap \mathcal{Q}_{\mu'_\bullet}) \leq \langle \rho, |\mu'_\bullet| - \lambda \rangle < \langle \rho, |\mu_\bullet| - \lambda \rangle.)$$

Proof of Proposition 3.1: By Theorem 2.1 we have

$$\mathcal{A}_{\mu_1} \star \cdots \star \mathcal{A}_{\mu_r} = \bigoplus_{\lambda \leq |\mu_\bullet|} \mathcal{A}_\lambda \otimes V^\lambda_{\mu_\bullet}.$$

Let $d = -\dim(\mathcal{Q}_\lambda)$. Now apply $\mathcal{H}^d_y$ (take the dth cohomology stalk at y) to both sides of this equation.

The right hand side gives the vector space $V^\lambda_{\mu_\bullet}$, since $\mathcal{A}_\lambda$ restricted to $\mathcal{Q}_\lambda$ is $\bar{\mathbb{Q}}_\ell[\dim(\mathcal{Q}_\lambda)]$ (the constant sheaf placed in degree $-\dim(\mathcal{Q}_\lambda)$) and since by definition of intersection complexes, $\mathcal{H}^d_y \mathcal{A}_{\lambda'}$ is zero if $\lambda < \lambda'$. (In general, $\mathcal{H}^i \mathrm{IC}(X)|_{X_\alpha} = 0$ if $i > -\dim(X_\alpha)$ and also for $i \geq -\dim(X_\alpha)$ if X_α is not the open stratum in X.)

By Lemma 2.3, on the left hand side we get

$$H^d(m_{\mu_\bullet}^{-1}(y), \mathrm{IC}(\widetilde{\mathcal{Q}}_{\mu_\bullet})).$$

The result now follows by virtue of the following lemma.

Lemma 3.2. *Let $f : X = \cup_\alpha X_\alpha \to Y = \cup_\beta Y_\beta$ be a semi-small morphism between proper stratified schemes, and suppose $y \in Y_\beta$. Let $d = -\dim(Y_\beta)$. Then there is a canonical isomorphism*

$$H^d(f^{-1}(y), \mathrm{IC}(X)) = \bar{\mathbb{Q}}_\ell^{C_{max}(y)},$$

where $C_{max}(y)$ is the set of irreducible components of $f^{-1}(y)$ having the maximal possible dimension $\frac{1}{2}(\dim(X) - \dim(Y_\beta))$.

Proof. Let X^0 denote the open (smooth) stratum of X; recall that $\mathrm{IC}(X)|_{X^0} = \bar{\mathbb{Q}}_\ell[\dim(X)]$. Suppose we fix $X_\alpha \neq X^0$. We claim that

$$H^d(f^{-1}(y) \cap X_\alpha, \mathrm{IC}(X)) = 0.$$

Indeed, if not, then the "local-global" spectral sequence

$$H^p(f^{-1}(y) \cap X_\alpha, \mathcal{H}^q \mathrm{IC}(X)) \Rightarrow H^{p+q}(f^{-1}(y) \cap X_\alpha, \mathrm{IC}(X))$$

shows that there exists q and $p = d - q$ such that the initial term is non-zero. We have therefore $q < -\dim(X_\alpha)$, which together with the semi-smallness of f gives

$$p \leq 2(\dim(f^{-1}(y) \cap X_\alpha)) \leq \dim(X_\alpha) - \dim(Y_\beta) < -q - \dim(Y_\beta),$$

which is clearly impossible since $p + q = d = -\dim(Y_\beta)$.

The same argument shows that $H^d(f^{-1}(y) \cap (X \backslash X^0), \mathrm{IC}(X)) = 0$.

Since $f^{-1}(y)$ and $f^{-1}(y) \cap (X \backslash X^0)$ are proper, their cohomology and compactly-supported cohomology agree. It follows from the above remarks that

$$H^d(f^{-1}(y), \mathrm{IC}(X)) = H_c^d(f^{-1}(y) \cap X^0, \bar{\mathbb{Q}}_\ell[\dim(X)]) = H_c^{d+\dim(X)}(f^{-1}(y) \cap X^0, \bar{\mathbb{Q}}_\ell),$$

from which the lemma follows easily. $\square$

4. Proof of Proposition 1.3, Part (2)

The next lemma follows from the definition of convolution in $\mathcal{H}_q$, and is left to the reader.

Lemma 4.1. $c_{\mu_\bullet}^\lambda(q) = \#(m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu_\bullet})(\mathbb{F}_q).$

Therefore, if f_λ occurs in $f_{\mu_1} * \cdots * f_{\mu_r}$, we obviously have $m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu_\bullet} \neq \emptyset$.

Conversely, suppose that $m_{\mu_\bullet}^{-1}(y) \cap \mathcal{Q}_{\mu_\bullet} \neq \emptyset$. It follows from this that $c_{\mu_\bullet}^\lambda(q^n) > 0$ for $n \gg 0$. Using the definition of convolution, this means that

$$t_\lambda \in K_{q^n} t_{\mu_1} K_{q^n} \cdots K_{q^n} t_{\mu_r} K_{q^n}.$$

But by the discussion below, this condition can be expressed purely in terms of the extended affine Weyl group, and in particular, it is independent of n. Therefore, it holds for $n = 1$ and thus $c_{\mu_\bullet}^\lambda(q) > 0$.

To complete the argument proving independence of n, let $\widetilde{W} = X_*(T) \rtimes W$ denote the extended affine Weyl group. Denote the translation element in $\widetilde{W}$ corresponding to $\mu \in X_*(T)$ also by the symbol t_μ. Let $I_{q^n} \subset K_{q^n}$ be the Iwahori subgroup defined to be the inverse image of B under the homomorphism $G(\mathbb{F}_{q^n}[[t]]) \to G(\mathbb{F}_{q^n})$ given by $t \mapsto 0$. For simplicity, write $K = K_{q^n}$ and $I = I_{q^n}$.

Then $K = IWI := \coprod_{w \in W} IwI$ and $G(\mathbb{F}_{q^n}((t))) = I\widetilde{W}I := \coprod_{w \in \widetilde{W}} IwI$. Furthermore, standard results for BN pairs yield the identity

$$ IxIyI \subset \coprod_{\tilde{y} \preceq y} Ix\tilde{y}I, $$

for $x, y \in \widetilde{W}$, where $\tilde{y}$ ranges over elements preceding y in the Bruhat order [1] on $\widetilde{W}$. Using this we see

$$ t_\lambda \in Kt_{\mu_1}K \cdots Kt_{\mu_r}K \Leftrightarrow t_\lambda \in IWt_{\mu_1}WI \cdots IWt_{\mu_r}WI $$
$$ \Leftrightarrow \exists\, x_i \in Wt_{\mu_i}W \text{ such that } t_\lambda \in Ix_1I \cdots Ix_rI. $$

But by standard facts for BN-pairs, the set $\mathcal{S} \subset \widetilde{W}$ appearing in the union

$$ Ix_1I \cdots Ix_rI = \coprod_{w \in \mathcal{S}} IwI $$

depends only on the elements $x_1, \ldots, x_r \in \widetilde{W}$ (and not on the power q^n in $I = I_{q^n}$).

5. End of proof of Theorem 1.1

It remains to prove the formula

$$ c^\lambda_{\mu_\bullet}(q) = \dim(V^\lambda_{\mu_\bullet})\, q^{\langle \rho, |\mu_\bullet| - \lambda \rangle} + \{\text{terms of lower } q\text{-degree}\}. $$

Clearly we may prove this after base extension (enlarging q), so that we may assume the irreducible components of $m^{-1}_{\mu_\bullet}(y) \cap \mathcal{Q}_{\mu_\bullet}$ are defined over $\mathbb{F}_q$. But then taking Lemma 4.1 and Proposition 3.1 into account, the formula follows immediately from the following lemma, itself a consequence of Deligne's paper [Weil2].

Lemma 5.1. *If X is a geometrically irreducible $\mathbb{F}_q$-variety, then the function $\#X(\mathbb{F}_q)$ is of the form $q^{\dim(X)} + r(q)$, where $|r(q)| \le O(q^{\dim(X) - 1/2})$.*

6. The P-R-V conjecture and Propositions 1.4 and 1.5

The Parthasarathy-Ranga-Rao-Varadarajan (P-R-V) conjecture has been proved independently by S. Kumar and O. Mathieu. See [Lit], section 10, for a short proof using the Littelmann path model.

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[1]The Bruhat order here is defined using the set of simple affine reflections through the hyperplanes bounding the *opposite* base alcove, i.e., the alcove $w_0(A)$, where A is the alcove fixed by I.

Theorem 6.1 (P-R-V Conjecture)**.** *For* $1 \leq i \leq r$*, let* μ_i *be a dominant coweight, and let* $w_i \in W$ *be an element of the finite Weyl group. Suppose* $\nu = w_1\mu_1 + \cdots + w_r\mu_r$ *is dominant. Then* V_ν *occurs with multiplicity at least one in the tensor product*

$$V_{\mu_1} \otimes \cdots \otimes V_{\mu_r}.$$

(This is usually stated only in the case $r = 2$, but the above version follows easily by induction on r.)

Theorem 6.1 is the main ingredient to proving Proposition 1.4 and the assertion **Rep**$(\mu_\bullet, \lambda)$ in Proposition 1.5. Indeed, for the latter we only need to verify the following lemma to be able to take $\nu = \lambda$ in Theorem 6.1.

Lemma 6.2. *Suppose that each dominant coweight* μ_i *is minuscule, and that* λ *is dominant and satisfies* $\lambda \leq \mu_1 + \cdots + \mu_r$*. Then there exist elements* $w_i \in W$ *(*$1 \leq i \leq r$*), such that*

$$\lambda = w_1\mu_1 + \cdots + w_r\mu_r.$$

For $V \in \mathrm{Rep}(\hat{G})$, let $\Omega(V) \subset X^*(\hat{T}) = X_*(T)$ denote the set of its weights with respect to the dual torus $\hat{T}$. Recall that

$$\Omega(V_\mu) = \{\nu \in X^*(\hat{T}) \mid w\nu \leq \mu, \ \forall w \in W\}.$$

The following well-known lemma is left to the reader.

Lemma 6.3. *Let* $\mu_1, \ldots, \mu_r$ *be dominant coweights (also viewed as weights for* $\hat{G}$*). Then*

$$\Omega(V_{|\mu_\bullet|}) = \Omega(V_{\mu_1} \otimes \cdots \otimes V_{\mu_r}) = \{\nu_1 + \cdots + \nu_r \mid \nu_i \in \Omega(V_{\mu_i}), 1 \leq i \leq r\}.$$

Note that any dominant λ satisfying $\lambda \leq |\mu_\bullet|$ is necessarily a weight for $V_{|\mu_\bullet|}$. Also, μ minuscule implies that $\Omega(V_\mu) = W\mu$. Thus, Lemma 6.3 implies Lemma 6.2.

In the same way, we get the following result, proving Proposition 1.4.

Proposition 6.4. *Suppose* $\lambda \leq |\mu_\bullet|$ *is dominant. Then there exist dominant coweights* $\mu'_i \leq \mu_i$ *and elements* $w_i \in W$ *(*$1 \leq i \leq r$*) such that* $\lambda = w_1\mu'_1 + \cdots + w_r\mu'_r$*. Consequently, the P-R-V conjecture implies that* **Rep**$(\mu'_\bullet, \lambda)$ *holds. In particular, we have*

$$\mathbf{Hecke}(\mu_\bullet, \lambda) \Longrightarrow \mathbf{Rep}(\mu'_\bullet, \lambda).$$

As for the assertion **Hecke**$(\mu_\bullet, \lambda)$ of Proposition 1.5, at this point one could appeal to Theorem 1.1, but we prefer to give a direct proof. Indeed, the assertion is obvious from the remark that $m_{\mu_\bullet}(\mathcal{Q}_{\mu_\bullet}) = \bar{\mathcal{Q}}_{|\mu_\bullet|}$. This in turn follows from the surjectivity of $m_{\mu_\bullet}$, since μ_i minuscule for every i means $\bar{\mathcal{Q}}_{\mu_i} = \mathcal{Q}_{\mu_i}$ and thus $\widetilde{\mathcal{Q}}_{\mu_\bullet} = \mathcal{Q}_{\mu_\bullet}$.

7. Spaltenstein-Springer varieties and Proposition 1.7

We begin with the statement of a crucial theorem of Spaltenstein [Sp]. Let V denote a k-vector space of dimension d, and let $(d_1, \ldots, d_r)$ denote an ordered r-tuple of non-negative integers such that $d_1 + \cdots + d_r = d$. The r-tuple $d_\bullet$ determines a standard parabolic subgroup $P \subset \mathrm{GL}(V)$. Let us consider the variety of partial flags of type P:

$$\mathcal{P} = \{V_\bullet = (V = V_0 \supset V_1 \supset \cdots \supset V_r = 0) \mid \dim(V_{i-1}/V_i) = d_i, \ 1 \le i \le r\},$$

which is isomorphic to $\mathrm{GL}(V)/P$.

For any nilpotent endomorphism $T \in \mathrm{End}(V)$, let $\mathcal{P}^T$ denote the closed subvariety of $\mathcal{P}$ consisting of partial flags $V_\bullet$ such that T stabilizes each V_i. This is simply the fiber over T of the partial Springer resolution

$$\pi : \widetilde{\mathcal{N}}^P \to \mathcal{N},$$

where $\mathcal{N} \subset \mathrm{End}(V)$ is the nilpotent cone, $\widetilde{\mathcal{N}}^P = \{(T, V_\bullet) \in \mathcal{N} \times \mathcal{P} \mid V_\bullet \in \mathcal{P}^T\}$, and the morphism π is the obvious forgetful one.

Inside $\mathcal{P}^T$ we may consider the closed subvariety $\mathcal{P}^T_{\min}$ consisting of partial flags $V_\bullet \in \mathcal{P}^T$ such that

$$T \text{ acts by zero on } V_{i-1}/V_i \ (1 \le i \le r).$$

We call such varieties $\mathcal{P}^T_{\min}$ the *Spaltenstein-Springer varieties*.

We have the following fundamental result of N. Spaltenstein ([Sp], final Corollary).

Theorem 7.1 (Spaltenstein). *The irreducible components of a Spaltenstein-Springer variety $\mathcal{P}^T_{\min}$ all have the same dimension.*

Now we turn to the proof of Proposition 1.7, which we will reduce to Theorem 7.1.

Recall that $\mu_\bullet = (\mu_1, \ldots, \mu_r)$, where each μ_i is a dominant minuscule coweight for GL_n. Working in the affine Grassmannian for GL_n and fixing $y \in \mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{|\mu_\bullet|}$, we want to show that all the irreducible components of $m_{\mu_\bullet}^{-1}(y)$ have the same dimension. Without loss of generality, we may assume $\mu_i = (1^{d_i}, 0^{n-d_i})$, where $1 \le d_i \le n-1$. Write $d := d_1 + \cdots + d_r$. Suppose the point $y \in \mathcal{Q}_\lambda$ corresponds to the $\mathcal{O}$-lattice $L \subset F^n$. We have $L \subset \mathcal{O}^n$. Let T denote the nilpotent endomorphism of the d-dimensional k-vector space $V := \mathcal{O}^n/L$ induced by multiplication by t. Then the fiber $m_{\mu_\bullet}^{-1}(y)$ consists of chains of k-vector spaces

$$\mathcal{O}^n = L_0 \supset L_1 \supset \cdots \supset L_r = L$$

satisfying

- $\dim(L_{i-1}/L_i) = d_i$, for each i,
- T preserves each L_i,
- T acts by zero on each L_{i-1}/L_i.

In other words, $m_{\mu_\bullet}^{-1}(y)$ is a Spaltenstein-Springer variety for the d-dimensional k-vector space V and the nilpotent operator T. Thus, Proposition 1.7 follows from Theorem 7.1.

8. Proofs of Proposition 1.8 and Theorem 1.2

Proof of Proposition 1.8: We need the following set-up in order to bring Proposition 1.7 into play. Write each coweight μ_i as a sum of dominant minuscule coweights

$$\mu_i = \nu_{i1} + \nu_{i2} + \cdots$$

We consider the following diagram

$$(\mathcal{Q}_{\nu_{11}} \widetilde{\times} \cdots) \widetilde{\times} \cdots \widetilde{\times} (\mathcal{Q}_{\nu_{r1}} \widetilde{\times} \cdots)$$
$$\downarrow \eta$$
$$\bar{\mathcal{Q}}_{\mu_1} \widetilde{\times} \cdots \widetilde{\times} \bar{\mathcal{Q}}_{\mu_r}$$
$$\downarrow m_{\mu_\bullet}$$
$$\bar{\mathcal{Q}}_{|\mu_\bullet|},$$

where $\eta = m_{\nu_{1\bullet}} \widetilde{\times} \cdots \widetilde{\times} m_{\nu_{r\bullet}}$. The composition $m_{\nu_{\bullet\bullet}} = m_{\mu_\bullet} \circ \eta$ is just the usual forgetful morphism, once the domain is identified with the twisted product of the varieties $\mathcal{Q}_{\nu_{ij}}$. In particular, all three morphisms $m_{\mu_\bullet}$, η, and $m_{\nu_{\bullet\bullet}}$ are semi-small, proper, birational, surjective, and locally trivial in the stratified sense. Moreover, the conclusion of Proposition 1.7 applies to $m_{\nu_{\bullet\bullet}}$. Note that $|\nu_{\bullet\bullet}| = |\mu_\bullet|$.

Let $y \in \mathcal{Q}_\lambda \subset \bar{\mathcal{Q}}_{\mu_\bullet}$. Suppose C' is an irreducible component of $m_{\mu_\bullet}^{-1}(y)$ such that $C' \cap \mathcal{Q}_{\mu'_\bullet}$ is an open dense subset in C', so that $\dim(C' \cap \mathcal{Q}_{\mu'_\bullet}) = \dim(C')$. Let C be any irreducible component of $\eta^{-1}(C')$ which dominates C'. Then $C \cap \eta^{-1}(\mathcal{Q}_{\mu'_\bullet})$ is non-empty, hence is an open dense subset of C. Thus, using the semi-smallness and local triviality of η, we get

$$\begin{aligned}
\dim(C) &= \dim(C \cap \eta^{-1}(\mathcal{Q}_{\mu'_\bullet})) \\
&\leq \dim(\eta^{-1}(C' \cap \mathcal{Q}_{\mu'_\bullet})) \\
&\leq \langle \rho, |\nu_{\bullet\bullet}| - |\mu'_\bullet| \rangle + \dim(C' \cap \mathcal{Q}_{\mu'_\bullet}) \\
&= \langle \rho, |\mu_\bullet| - |\mu'_\bullet| \rangle + \dim(C').
\end{aligned}$$

Now C is an irreducible component of $m_{\nu_{\bullet\bullet}}^{-1}(y)$, hence by Proposition 1.7 has dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$. Therefore

$$\dim(C') \geq \langle \rho, |\mu'_\bullet| - \lambda \rangle,$$

and the desired equality follows from the semi-smallness of $m_{\mu_\bullet}$.

Proof of Theorem 1.2:

This follows immediately from Proposition 1.3, Proposition 1.8 and the remark that since $\mathcal{Q}_{\mu_\bullet}$ is open in $\widetilde{\mathcal{Q}}_{\mu_\bullet}$, any irreducible component C of the fiber $m_{\mu_\bullet}^{-1}(y)$ which meets $\mathcal{Q}_{\mu_\bullet}$ has $C \cap \mathcal{Q}_{\mu_\bullet}$ as a dense open subset. This completes the proof of Theorem 1.2.

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