

Calculus 131, section 8.1 The Trapezoidal Rule & Simpson's Rule

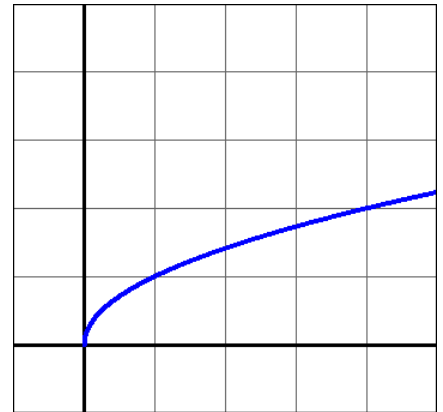
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Back in Math 130, section 7.3, you did Riemann sums. The interval $a \leq x \leq b$ was split up into n subintervals, called partitions, of width $= \frac{b-a}{n} = \Delta x$. Then a series of rectangles was drawn, each with a width of Δx and a height of $y = f(x)$. In section 8.1 we're only going to consider the rectangles whose height is in the middle of the interval—the *Midpoint Rule*:

$$\begin{aligned} \text{Area under the curve} &\cong f(x_1) * \Delta x + f(x_2) * \Delta x + f(x_3) * \Delta x + \dots + f(x_n) * \Delta x \\ &= \frac{b-a}{n} * [f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n)] \end{aligned}$$

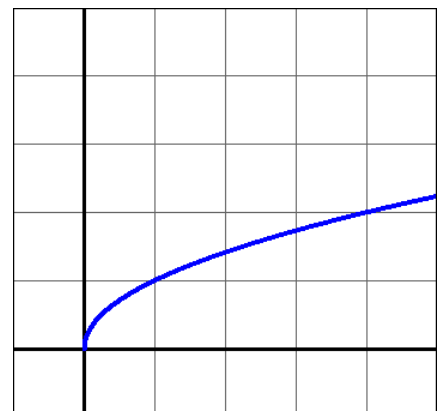
This formula (a *Riemann* sum) provides an approximation to the area under the curve for functions that are non-negative and continuous.

Example A, Midpoint Rule: Approximate the area under the curve $y = \sqrt{x}$ on the interval $2 \leq x \leq 4$ using $n = 5$ subintervals.



Recalling that “area under the curve from a to b ” $= \int_a^b f(x) dx$, the Midpoint Rule can be used to approximate a definite integral. Next, however, instead of rectangles, we're going to create a series of trapezoids and calculate areas. Each trapezoid will still have a width of Δx , but will also have *two* heights—one to each side, left and right, $f(x_{k-1})$ & $f(x_k)$. The formula for the area of a trapezoid is $[f(x_{k-1}) + f(x_k)] * \frac{\Delta x}{2}$.

Example A, Trapezoidal Rule: Approximate the area under the curve $y = \sqrt{x}$ on the interval $2 \leq x \leq 4$ using $n = 5$ subintervals. That is, approximate the definite integral $\int_2^4 \sqrt{x} dx$ by the Trapezoidal Rule.



We can take this example and generalize into a Trapezoidal Rule for n subintervals:

$$\int_a^b f(x) dx \cong f(x_0) * \frac{\Delta x}{2} + 2f(x_1) * \frac{\Delta x}{2} + 2f(x_2) * \frac{\Delta x}{2} + 2f(x_3) * \frac{\Delta x}{2} + \dots + 2f(x_{n-1}) * \frac{\Delta x}{2} + f(x_n) * \frac{\Delta x}{2}$$

$$\int_a^b f(x) dx \cong \frac{b-a}{n} * \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + f(x_3) + \dots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right]$$

Example A, integration: $\int_2^4 \sqrt{x} dx$.

$$\int_2^4 \sqrt{x} dx = \int_2^4 x^{1/2} dx = \frac{2}{3} x^{3/2} \Big|_2^4 = \frac{2}{3} \left(4^{3/2} - 2^{3/2} \right) = \frac{2}{3} \left(8 - 2^{3/2} \right).$$

The exact value for the area under the curve $y = \sqrt{x}$ on the interval $2 \leq x \leq 4$ is $\frac{2}{3} \left(8 - \sqrt{2^3} \right)$ which is approximately 3.44771525.

Each of the first two approximation methods, Midpoint Rule and Trapezoidal Rule, were each off by a bit. Also note that, as is usually the case with functions of the type we'll be seeing, while one of them is a bit high (our Midpoint Rule by about 0.000688768), the other is a bit low (our Trapezoidal Rule by about 0.00137879). Simpson's Rule combines this notion into a formula which weights the two according to their relative errors:

$$\int_a^b f(x) dx \cong \frac{2M + T}{3}.$$

However, it is unnecessary to calculate both Midpoint and Trapezoidal Rules before calculating Simpson's Rule. The comprehensive version of Simpson's Rule can be found by inserting the formulae for Midpoint and Trapezoidal Rules into $\frac{2M+T}{3}$ and simplifying.

$$\int_a^b f(x) dx \approx \frac{b-a}{3n} * [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

In this schema for explaining Simpson's Rule, successive values for x_k are alternately boundaries of subintervals (from Trapezoidal Rule—even subscripts) and midpoints of subintervals (from Midpoint Rule—odd subscripts). [Note that for Simpson's Rule, n must be an even number.]

Side note: Why do the "Midpoint Rule" terms have a coefficient of 4? During the substitution into $\frac{2M+T}{3}$ and subsequent simplification process these terms are multiplied << times 2 in the formula, and times 2 again to get a common denominator with the Trapezoidal Rule >>.

Example A, Simpson's Rule: Approximate the definite integral $\int_2^4 \sqrt{x} dx$ by Simpson's Rule using first $n = 4$ then $n = 8$ subintervals, and compare with results from Midpoint Rule, Trapezoidal Rule and Integration.

Trapezoidal Rule	Simpson's ($n = 4$)	Simpson's ($n = 8$)	Integration	Midpoint Rule
3.44633646	3.44769755	3.44771409	3.44771525	3.44840393

Example B: Approximate $\int_2^4 \frac{x}{x-1} dx$, using Midpoint Rule, Trapezoidal Rule and Simpson's Rule ($n = 4$).

answers: ≈ 3.08975469 , ≈ 3.11666667 , ≈ 3.1

Midpoint Rule with 4 subintervals, values of x_k are midpoints of subintervals

	x_1	x_2	x_3	x_4
x				
$f(x)$				

Trapezoidal Rule with 4 subintervals, values of x_k are endpoints of subintervals

	x_0	x_1	x_2	x_3	x_4
x					
$f(x)$					

Simpson's Rule, for $n = 4$, values of x_k are the same as for the Trapezoidal Rule above.

	x_0	x_1	x_2	x_3	x_4
x					
$f(x)$					

Example C: Approximate $\int_1^3 e^{x^2} dx$ using Simpson's Rule for $n = 8$. $\Delta x = ?$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x									

Note the very steep slope of the function $y = e^{x^2}$ when $x > 1$. At $x = 3$, $e^{3^2} = e^9 \cong 8103.08$, so an area under the curve of 1475.429234 makes sense.

Final note: While the explanations used strictly non-negative functions, the beauty of Simpson's Rule is that it applies to all continuous functions, whether or not they are non-negative.