Calculus 130, section 8.3 Average Value

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In Math 130, chapters 3 through 6 used a function that represented an *amount*. The derivative (= slope of the curve) gave a *rate of change*. In chapter 7, using a *rate of change* and an integral (= area under the curve) you were able to determine an *amount*. There are, however, other uses of integrals.

Consider the average of a group of numbers: average $= \bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$. The average value of a function is calculated in a similar manner: average of a function $= \bar{y} = \dfrac{f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)}{n}$.

However, a continuous function consists not of discrete values which can be added together, but an infinite number of values. A little algebraic manipulation will help us find a way to calculate this infinite sum. First we "raise to higher terms" by multiplying numerator and denominator by the same number. We choose the number $(b - a)$.

$$\text{average of a function } = \bar{y} = \frac{f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)}{n} * \frac{b-a}{b-a}$$

Rearranging using the commutative and associative properties:

$$\text{average of a function } = \bar{y} = \frac{1}{b-a} * \left[f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)\right] * \frac{b-a}{n}$$

The latter part of the formula should look familiar: it's the Riemann sum for the area under the curve. Thus,

$$\text{average of a function } = \bar{y} = \frac{1}{b-a} * \lim_{\Delta x \to 0} \left[f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)\right] * \Delta x = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Example A: Find the average value of the function $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *answer*: $\frac{2}{\pi}$

Example B: Find the average value of the function $f(x) = xe^{2x}$ on the interval $[0, 2]$. *answer*: $\frac{3}{8}e^4 + \frac{1}{8}$

Example C: Blood flows through an artery fastest at the center and slowest next to the artery wall. In 1842 the French physician Poiseuille developed an equation for the rate of blood flow in an artery with radius 0.2 cm: $v(x) = 40 - 990x^2$ where v is in centimeters per second and x = distance from the center in centimeters. (The equation will be different for an artery of different length and if blood pressure is not normal.) What is the average velocity of the blood flow through the artery? *answer*: 26.8 cm/sec

Example D: Find the average value of the function $f(x) = x^2 e^{x^3}$ on the interval [0, 1]. *answer*: $\frac{1}{3}e - \frac{1}{3}$