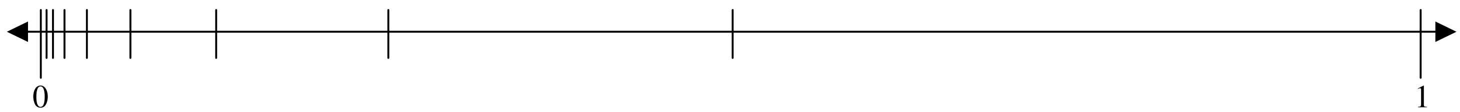
Calculus 131, section 8.4 Improper Integrals

notes prepared by Tim Pilachowski

We now get to do a more thorough examination of the concept of limits in mathematics. Non-technically, taking a limit is moving constantly toward something without ever getting there. Finding $\lim\limits_{x \to \infty}$ is akin to walking toward the horizon: even though you keep moving, there is always more horizon off in the distance.

Here's another perspective: Consider decreasing your distance away from an object by half, then half again, then half again, etc. This is like being on a number line at 1, and moving toward 0.



First you'd go to $\frac{1}{2}$, then $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^{100}}, \ldots, \frac{1}{2^{1000}}, \ldots, \frac{1}{2^{1,000,000}}, \ldots, \frac{1}{2^{1,000,000,000,000,000}}, \ldots,$

You'd be always getting closer to 0, but never actually reaching 0. In mathematical parlance this would be finding $\lim\limits_{x \to 0}$.

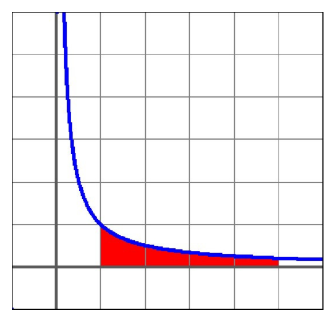
You've already encountered limits in Calculus I. (In the Math 130-131 text, see section 3.1.)

The definition of the first derivative is a limit (chapter 3.4): $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

A definite integral is a limit applied to a Riemann sum (chapter 7.3):

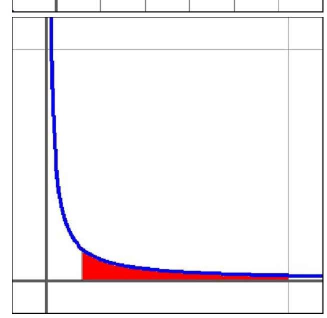
$$\lim_{\Delta x \to 0} \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right] * \Delta x = \int_a^b f(x)\, dx .$$

Two basic but very important principles will apply to the Examples below:: As a denominator → ∞, a fraction → 0, while any constants → themselves.

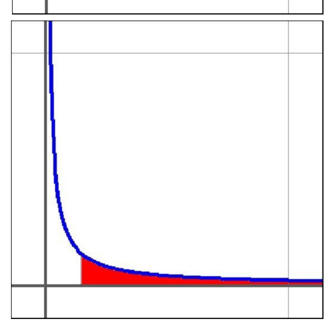


Example A, part 1: Find the area under the curve $y = \frac{1}{x}$ for $1 \le x \le 5$.

answer: ln 5



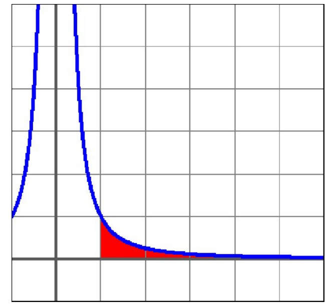
Example A, part 2: Find the area under the curve $y = \frac{1}{x}$ for $1 \le x \le b$ for some positive number $b > 1$. *answer*: $\ln b$



Example A, part 3: Find the area under the curve $y = \frac{1}{x}$ for $1 \le x < \infty$.

answer: The area under this curve cannot be assigned a number value.

Intuition is not enough, however. In other cases, what appears to be an "infinite" area actually converges to a particular limit.



Example B: Find the area under the curve $y = \dfrac{1}{x^2}$ for $1 \le x < \infty$.

answer: converges to 1

Example C. $\int_3^\infty \dfrac{x}{\sqrt{x^2-2}}\,dx$ *answer*: divergent

Example D. $\int_3^\infty \dfrac{x}{\left(x^2-2\right)^2}\,dx$ *answer*: converges to $\frac{1}{14}$

Example E. $\int_0^\infty xe^{-2x}\,dx$ *answer*: converges to $\frac{1}{4}$

Examples F. Like previous experience with integration, you'll need to choose the best method.

1) $\int_1^\infty \frac{10}{x^2}\,dx$ *answer*: converges to 10

2) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ *answer*: converges to 1

3) $\int_e^\infty \frac{\ln x}{x^2}\,dx$ *answer*: converges to $\frac{2}{e}$