Calculus 131, section 9.1 Functions of More Than One Variable

notes by Tim Pilachowski

In your algebra and calculus up to this time, the functions you have dealt with have mostly been functions of one variable, like $f(x) = x^2 - x + 1$. Pick an x, plug it in, and calculate. But you have worked with functions of more than one variable, even if they were not called that at the time.

Consider the basic formulas from geometry. Some, like those for circles, are functions of one variable. We could write area of a circle as a function of radius: $A(r) = \pi r^2$. The rectangle formulas require two variables, one for each dimension. If we let x = length and y = width, then the perimeter formula is $P(x, y) = 2x + 2y$. The area formula would be $A(x, y) = xy$.

Those of you who took Math 113 may remember the objective function from linear programming, which was a function of two variables, something like $z = 4x + 7y$.

Example A: Exponential growth and decay (section 2.3) could have been written in function notation: $y(y_0, k, t) = y_0e^{kt}$. (Note that e is not a variable but the number called Euler's number.) One, two or all three variables may change from one scenario to the next:

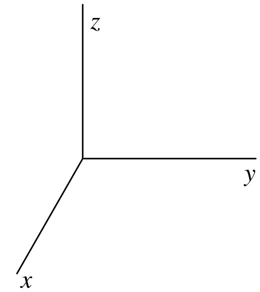
y_0	k	t	$y(y_0, k, t) = y_0e^{kt}$ exact value	approximate value

Example B: Section 6.4 refers to velocity of blood in a blood vessel, $v = k\left(R^2 - r^2\right)$, where R = radius of the blood vessel, r is the distance of the flowing blood from the center, and k is a constant of proportionality. At the beginning of her run, a skier's blood vessel has contracted to a radius of 0.06 mm. As she skis she warms up and the blood vessel expands to its usual radius of 0.08 mm. Compare the average velocity of blood flowing through the vessel at the two times. Assume k is 375.
answers: 0.9 mm/min, 1.6 mm/min

Example C: Your text, in Example 8, then again in exercise 34, introduces a class of functions that looks like the Cobb-Douglas production function from economics. The surface area S of a human being, in m^2, is approximated by $S(W, H) = 0.202W^{0.425}H^{0.725}$, where W = weight in kg and H = height in m. Given my height of 1.83 m and ideal weight of 85 kg, calculate my ideal surface area. (In exercise 34, you'll need to use the conversion factors provided.) *answer*: ≈ 2.068 m^2

A function of two variables, $z = f(x, y)$, can be graphed on a three-dimensional grid. Picture the corner of a room where the length and width of the floor form the x-axis and y-axis. The vertical where the two walls meet is the z-axis. Where the Cartesian 2-D grid is split into 4 quadrants, our 3-D grid is divided into 8 octants.

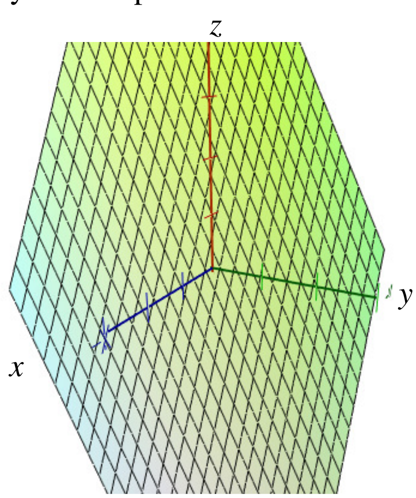
Example D: Graph the first-octant portion of the plane $4x + 2y + 3z = 12$.

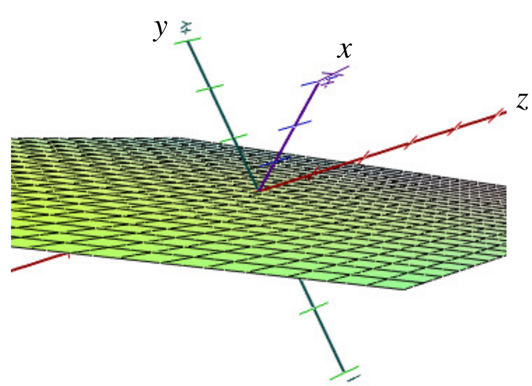


The “standard” orientation, while helpful from a conceptual view, will not always provide a clear picture of the function. A three-dimensional shape translated into two dimensions can obscure the true nature of the shape. It would be nice to be able to rotate the picture through the three dimensions to get a clearer idea.

A very good, and free, graphing calculator for your computer can be downloaded from www.graphcalc.com. This utility will allow you to graph both two- and three-dimensional graphs, and rotate the three-dimensional graphs either automatically or manually. (It’s the utility I used for the examples below.) It also has the ability to do some evaluations, as well as to graph parametric and polar equations. (You have your choice of colors to draw the graph in, too.)

Example E: Graph and explore the rectangle perimeter function $z = 2x + 2y$.
The graph shown to the left below has been drawn on the axes as oriented above, but it’s hard to tell what shape it really has. The graph to the right has been rotated to illustrate more clearly that the shape is a flat surface—in geometry terms a plane.

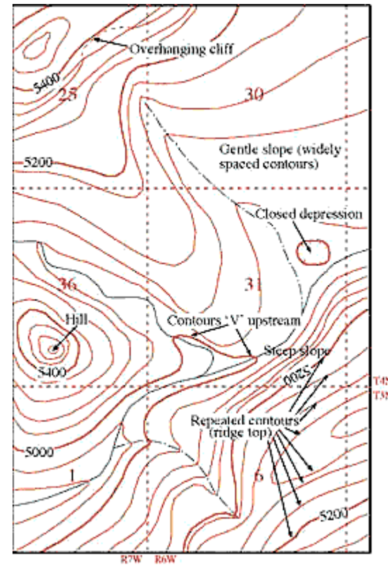




Let’s go back briefly to the type of problems we’ve encountered before this. Even when we needed to use a function of more than one variable, there was enough information given so that we could do some sort of substitution and turn it into an equation with one variable that we could solve.

Example F: Farmer Bob has a rectangular corral and 120 feet of fencing. Write an equation in one variable that represents the area of the corral. *Answer*: $A = x(60 - x)$

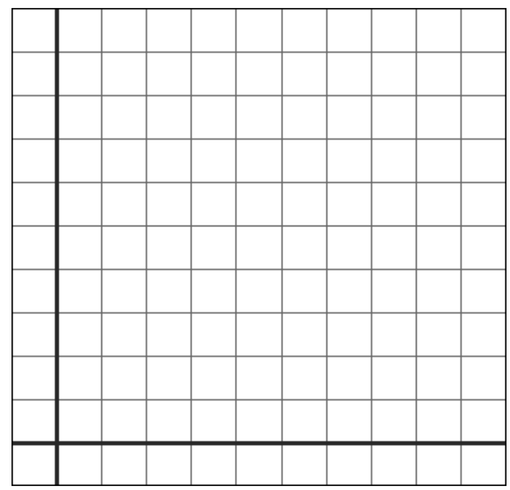
A similar process provides a means of taking a three-dimensional object and considering it in two dimensions. The concept is the same one used in drawing topographical maps that show the elevations of terrain. The flat surface is laid out in latitude and longitude (the x and y) while a series of curves show the elevation (z) of the terrain. Tracing along a curve shows the places on the ground that all have the same elevation. Widely spaced curves indicate a gentler slope. Curves close together indicate a steep slope.



If we choose a series of values for z, the result is a series of equations in x and y that can be graphed as usual on the Cartesian grid. These are called *level curves*. Your text exercises ask you to graph the first octant. Some of my examples use the whole Cartesian grid to give you a more comprehensive picture.

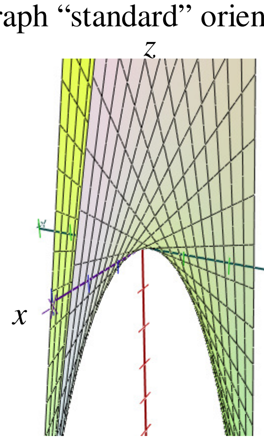
Example E revisited: Draw level curves for $z = 2x + 2y$ for $z = 10, 8, 6, 4$, and 2.
Given $z = 10$, then $10 = 2x + 2y \Rightarrow 10 - 2x = 2y \Rightarrow 5 - x = y$.
The others are found in a similar fashion.

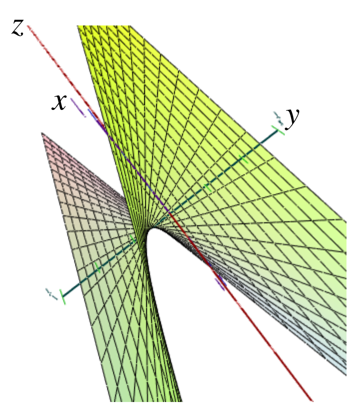
z	$z = 2x + 2y$	$f(x) = y = \ldots$
10	$10 = 2x + 2y$	$y = 5 - x$



Each level curve on the grid represents a different value for z.
The flat surface of $z = 2x + 2y$ rises at an angle as z increases.

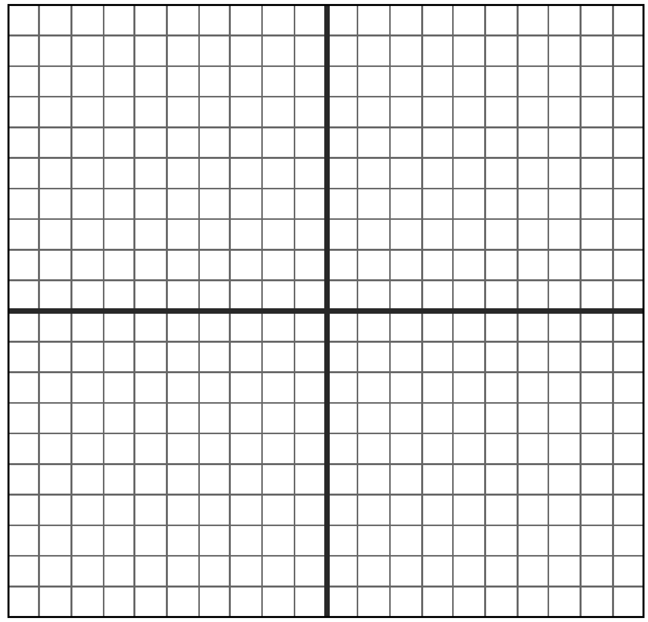
Example G: Graph and explore the rectangle area function $z = xy$. Sketch the level curves for $z = 10, 8, 6, 4, 2$.
The graph "standard" orientation of the x-y-z axes is on the left, and a rotation of the graph on the right.



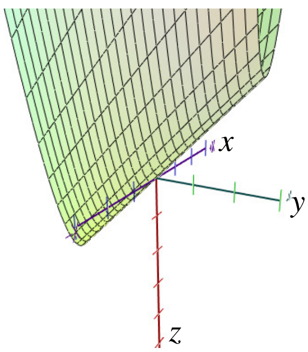


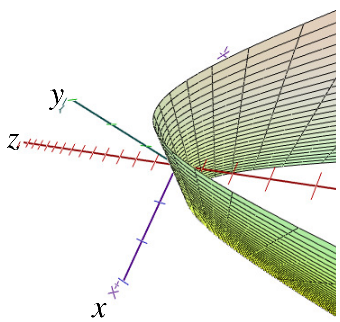
The "twist" (called a "saddle point") in the 3-D surface is easier to see in the rotated version. The symmetric nature of the twist centered at the origin is even more apparent from the level curves.

z	$z = xy$	$f(x) = y = \ldots$



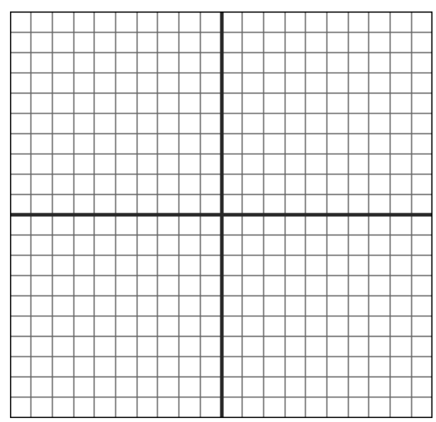
Example H: Draw the 3-D graph and level curves for the function $z = x^2 - y$.





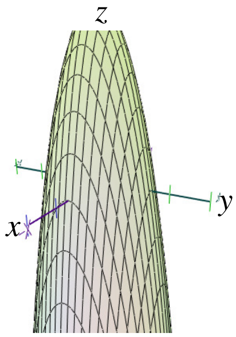
The orientation on the left shows a hint of the parabolic nature of the surface, which becomes clearer in the rotation. The level curves make it obvious.

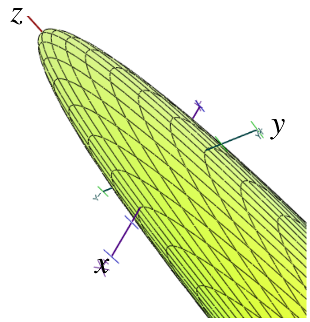
z	$z = x^2 - y$	$f(x) = y = \ldots$



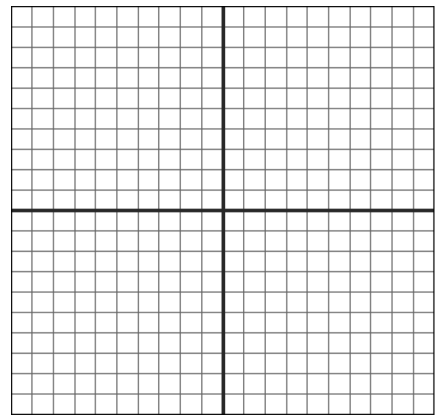
Something we'll be interested in later on is developing a mathematical process to determine the location and nature of the minimum of the shape.

Example I: Draw the 3-D graph and level curves for the function $z = 9 - x^2 - y^2$.





The surface on the left shows something of the curve, but the top is outside the viewing window. Tilting the graph slightly allows us to see the maximum. The level curves show the circular nature of the shape. At a later time we'll be developing a method of finding the location of that absolute maximum.



z	$z = 9 - x^2 - y^2$	equation in x and y

Example I revisited: Given $f(x, y) = 9 - x^2 - y^2$, find $\dfrac{f(x+h, y) - f(x, y)}{h}$. answer: $-2x - h$