Calculus 131, section 9.2 Partial Derivatives

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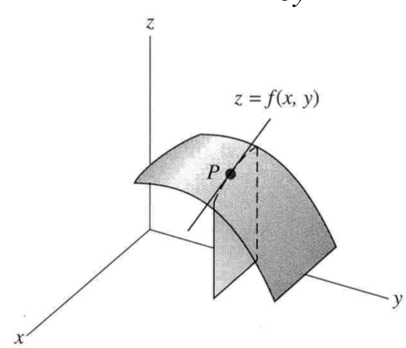
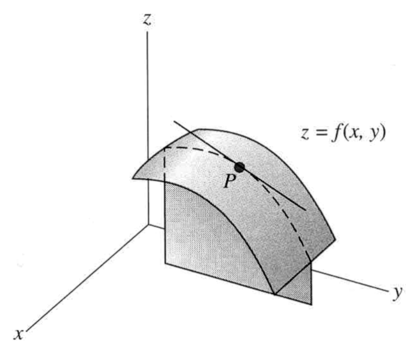
When working with functions of more than one variable, the question in calculus becomes: how can we evaluate the rate of change? The answer is called a *partial derivative*. Given a function $f(x, y)$ or $f(x, y, z)$, the partial derivative of f with respect to x, $\frac{\partial f}{\partial x} = f_x$, is found by treating all variables other than x as constants. Technically, we're finding $\lim_{h \to 0} \frac{f(x+h,\, y) - f(x,\, y)}{h}$, $\lim_{h \to 0} \frac{f(x+h,\, y,\, z) - f(x,\, y,\, z)}{h}$, etc. The partial derivatives $\frac{\partial f}{\partial y} = f_y$ and $\frac{\partial f}{\partial z} = f_z$ have analogous definitions.

Example A: Given the function $f(x,\, y) = 9 - x^2 - y^2$ find $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$. *answers*: $-2x$; $-2y$

A geometric interpretation of partial derivative is pictured below. In each figure $f(x, y)$ is the curved surface. In the figure on the left, with y treated as a constant, the tangent line goes the same general direction as the x-axis, and $\frac{\partial f}{\partial x}$ is the slope of that tangent at point P. With x treated as a constant, the tangent line goes the same general direction as the y-axis, and $\frac{\partial f}{\partial y}$ is the slope of that tangent at point P.





Example B: Given the function $f(x,\, y,\, z) = x^2 yz - ze^{xy} + \frac{x}{y}\ln(z)$ find $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$ and $\frac{\partial f}{\partial z} = f_z$.

answers: $2xyz - yze^{xy} + \frac{1}{y}\ln(z)$; $x^2 z - xze^{xy} - \frac{x}{y^2}\ln(z)$; $x^2 y - e^{xy} + \frac{x}{yz}$

Example B (continued)

Example C: For $f(x,\ y) = x^2 + 3xy + y + 7$, determine $\frac{\partial f}{\partial x}(5,\ 8) = f_x(5,\ 8)$ and $\frac{\partial f}{\partial y}(5,\ 8) = f_y(5,\ 8)$.

answers: 34; 16

Example D: Given $z = x^2 + 4xy + y^2 - 12y$, find values of x and y such that both $z_x = 0$ and $z_y = 0$.
answer: $(x, y) = (4, -2)$

Example E: The surface area S of a human being, in m^2, is approximated by $S(W,\ H) = 0.202W^{0.425}H^{0.725}$, where W = weight in kg and H = height in m. Given my height of 1.83 m and ideal weight of 85 kg, describe how my surface area would be changing if I started to gain weight above my ideal.
answer: ≈ 0.0103 m^2 per kg gained

Example F: For $f(x,\ y) = \ln|2x+3y| + \sin(xy)$ determine $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial^2 f}{\partial x\,\partial y} = f_{yx}$, $\frac{\partial^2 f}{\partial y\,\partial x} = f_{xy}$.

Just like regular derivatives, higher order partial derivatives can be found. The first two are called respectively the second partial derivative with respect to *x* and the second partial derivative with respect to *y*. The second two are often called "mixed partial derivatives". In this case, as with most functions, the two mixed partials are equal. (You should use this fact to check your answers.) Note the use of the chain rule for both first and second partial derivatives. Note that the product rule is needed for the second partial derivatives.

answers: $f_{xx} = -4(2x+3y)^{-2} - y^2\sin(xy)$, $f_{yy} = -9(2x+3y)^{-2} - x^2\sin(xy)$,

$f_{xy} = f_{yx} = -6(2x+3y)^{-2} - xy\sin(xy) + \cos(xy)$