Calculus 131, section 9.5 Double Integrals

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Example A: Evaluate $\int x^2 + 4xy + y^2 - 12y + 2\; dx$.

The principle here is the same as for finding a partial derivative with respect to *x*: all other variables are treated like constants. *answer*: $\frac{1}{3}x^3 + 2x^2y + xy^2 - 12xy + 2x + C$

Do-it-yourself example: $\int x^2 + 4xy + y^2 - 12y + 2\; dy = x^2y + 2xy^2 + \frac{1}{3}y^3 - 6y^2 + 2y + C$

Example A extended: Evaluate $\int_0^1 \int_1^2 x^2 + 4xy + y^2 - 12y + 2\; dx\, dy$.

The principle here is the same as for finding a mixed partial (second) derivative with respect to *x* then with respect to *y*. *answer*: $\frac{5}{3}$

Do-it-yourself version: $\int_1^2 \int_0^1 x^2 + 4xy + y^2 - 12y + 2\; dy\, dx$ yields the same result.

Example B: Find the double integral $\iint_R x\sqrt{2x^2 + 3y}\; dy\, dx$ over the rectangular region $0 \le x \le 1$, $1 \le y \le 2$.

This is really the same type of integral as in Example A extended. Be careful that you put the boundaries of integration into the correct integral. *answer*: $\frac{1}{45}\left(8^{5/2} - 5^{5/2} - 6^{5/2} + 3^{5/2}\right) \approx 1.1672$

Two notes on Example B:

1) We could have switched this around to do the more involved integration by substitution (*dx*) first, as long as we kept the boundaries of integration straight.

$$\int_0^1 \int_1^2 x\left(2x^2+3y\right)^{1/2}\, dy\, dx = \int_1^2 \int_0^1 x\left(2x^2+3y\right)^{1/2}\, dx\, dy$$

However, it turns out to be a good bit harder if we do the switch. Rule of thumb: Do the easier integration first.

2) Integration by parts may show up in text exercises, but since it's an even more labor-intensive process than substitution, we won't have time to do an example in lecture.

Example C: Find the volume under the surface $z = \dfrac{1}{xy}$ and above the rectangle $1 \le x \le e$, $1 \le y \le e^2$. *answer*: 2

This is really the same type of integral as in Example A extended and Example B. In other words, your exam could ask the same question in any one of these three ways.

Example D: Evaluate $\int_0^1 \int_{-x}^{x^2} x^2 + 3xy + 2y\, dy\, dx$. *answer*: $\frac{23}{120}$

Text practice exercise #60 is much like example D, except that you have to work for the boundaries of integration. The given equations $y = x$ and $y = 2x$ will be the boundaries of the integration dy. The boundaries of the integration dx will be the intersection of x and $2x$ (you can figure this one out for yourself) and $x = 1$ (the vertical line given to you).

New topic: Recall, for a two-dimensional function, average value for a curve $= \overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

When we expand to three dimensions, we can find "average value for a surface above a rectangular region" by manipulating the formula above to find average value across both x- and y-directions, i.e. using a double integral.

$$\overline{f(x,\ y)} = \frac{1}{b-a} * \frac{1}{d-c}\int_c^d \int_a^b f(x,\ y)\,dx\,dy$$

Note that the denominator of the fraction is the area of the given rectangular region which is below the surface (the function's graph). Your text expands this process to develop a formula $\overline{f(x,\ y)} = \frac{1}{A}\int_c^d \int_a^b f(x,\ y)\,dx\,dy$ where A is the area of any shape region. In this class, we'll stick to "average value for a surface above a rectangular region".

Example E: Find the average value of $z = e^{2x-3y}$ over the region R, bounded by $0 \le x \le 1$, $0 \le y \le 2$.
answer: $\frac{1}{12}\left(-e^{-4} + e^{-6} + e^{2} - 1\right) \approx 0.5311$

Bonus Example: Find the volume under the surface $z = \frac{1}{x+y}$ and above the rectangle $1 \le x \le e$, $1 \le y \le e$.

Set up the necessary double integral: $\int_1^e \int_1^e \frac{1}{x+y}\, dx\, dy$.

Integrate with respect to x using substitution: $u = x + y, \quad du = dx$.

$$\int_1^e \int \frac{1}{u}\, du\, dy = \int_1^e \ln u\, dy \quad \Rightarrow \quad \int_1^e \int_1^e \frac{1}{x+y}\, dx\, dy = \int_1^e \left[\ln(x+y)\right]_1^e dy = \int_1^e \left[\ln(e+y) - \ln(1+y)\right] dy$$

(We won't ask you to do something like the rest of this example on an exam – it requires a clever use of integration by parts that you don't need to know, as well as a method called "partial fractions" that you also do not need to know.)

The next few lines apply integration by parts to a generic model, using c, which is then used below where "c" is replaced by e and 1 respectively.

Rewrite $\int \ln(c+y)\, dy = \int \ln(c+y) * 1\, dy$,

then use integration by parts: $dv = 1\, dy$, $v = y$, $u = \ln(c+y)$, $\quad du = \frac{1}{c+y}\, dy$.

So $\int \ln(c+y)\, dy = \ln(c+y) * y - \int \frac{y}{c+y}\, dy$

(Here's where we use the fact that $\frac{y}{c+y} = 1 - \frac{c}{c+y}$. This is the "partial fractions" part.)

$$\begin{aligned} \int \ln(c+y)\, dy &= y\ln(c+y) - \int 1 - \frac{c}{c+y}\, dy \\ &= y\ln(c+y) - y + c\ln(c+y) \end{aligned}$$

Next we apply this result to our original integration.

$$\int_1^e \ln(e+y) - \ln(1+y)\, dy = \left[y\ln(e+y) - y + e\ln(e+y)\right]_1^e - \left[y\ln(1+y) - y + 1\ln(1+y)\right]_1^e$$

From here on out it's a lot of algebraic evaluation.

$$\int_1^e \ln(e+y) - \ln(1+y)\, dy$$

$= \left[e\ln(e+e) - e + e\ln(e+e)\right] - \left[1\ln(e+1) - 1 + e\ln(e+1)\right] - \left[e\ln(1+e) - e + 1\ln(1+e)\right] + \left[1\ln(1+1) - 1 + 1\ln(1+1)\right]$

$= \left[e\ln(2e) - e + e\ln(2e)\right] - \left[\ln(e+1) - 1 + e\ln(e+1)\right] - \left[e\ln(1+e) - e + \ln(1+e)\right] + \left[\ln(2) - 1 + \ln(2)\right]$

distributing the subtraction:

$= e\ln(2e) - e + e\ln(2e) - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$

using logarithm properties:

$= e\ln(2) + e\ln(e) - e + e\ln(2) + e\ln(e) - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$

$= e\ln(2) + e - e + e\ln(2) + e - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$

using commutativity of addition then combining like terms:

$= e - e + e + e + e\ln(2) + e\ln(2) - e\ln(e+1) - e\ln(1+e) - \ln(e+1) - \ln(1+e) + \ln(2) + \ln(2) + 1 - 1$

$= 2e + 2e\ln(2) - 2e\ln(e+1) - 2\ln(e+1) + 2\ln(2)$