Calculus 131, section 10.1   Gauss-Jordan Algorithm

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In the Math 131 text, the description of the Gauss-Jordan method tries to get upper and lower triangles with entries equaling 0 first, then multiplying rows as needed to get diagonal entries equaling 1. The algorithm described below accomplishes these goals in the opposite order.

For an augmented matrix representing a system of n equations in n variables:
Identify the diagonal, cells [row 1, col 1], [row 2, col 2], … ,[row n, col n]. If any diagonal entry = 0, interchange rows so that each diagonal entry ≠ 0.
Goals: A) in diagonal, all entries = 1
B) in triangles both above and below the diagonal, all entries = 0.
Note: To make the process flow a little more smoothly, you may want to multiply rows by constants to clear decimals and fractions, and (if possible) interchange rows so that entry in [row 1, col 1] = 1.

Gauss-Jordan algorithm: Repeat for each column Φ = 1, 2, … , n.
subroutine A: If needed, multiply row Φ by reciprocal of entry in [row Φ, col Φ].   (Goal A)
subroutine B: For all rows R_i above and below row Φ where [row i, col Φ] ≠ 0,
do R_i + (–1) [entry in row i, col Φ] times R_Φ.   (Goal B)
(Row working on) + (–1) (number that's there now) (Row of diagonal under consideration)

Illustration:

$$\begin{cases} 3x+y+z=6 \\ x+y-z=2 \\ 2x+y+2z=6 \end{cases} \text{ is written as a } 3\times 4 \text{ augmented matrix } \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 2 & 6 \end{array}\right].$$

Interchanging R_1 and R_2 makes [row 1, col 1] = 1. The Gauss-Jordan method becomes a tiny bit easier.

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 2 & 6 \end{array}\right] \Rightarrow \begin{array}{c} R_2 \to R_1 \\ R_1 \to R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & 1 & 1 & 6 \\ 2 & 1 & 2 & 6 \end{array}\right]$$

Φ = 1, subroutine A: not needed, since entry in [row 1, col 1] = 1

Φ = 1, subroutine B: goal is to get 0s above and below diagonal entry.

i = 2, Φ = 1: entry in [row 2, col 1] = 3, do row operation $R_2 + (-1)(3)R_1$:

$$\begin{array}{cccc} 3 & 1 & 1 & 6 \\ -3 & -3 & 3 & -6 \\ \hline 0 & -2 & 4 & 0 \end{array}$$

(working on Row 2)
(number now in row 2, col 1)
(diagonal row 1, col 1 under consideration)

$i = 3, \Phi = 1$: entry in [row 3, col 1] = 2, do row operation $R_3 + (-1)(2)R_1$:

$$\begin{array}{ccc:c} 2 & 1 & 2 & 6 \\ -2 & -2 & 2 & -4 \\ \hline 0 & -1 & 4 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & 1 & 1 & 6 \\ 2 & 1 & 2 & 6 \end{array}\right] \begin{array}{l} \\ R_2 + (-3)R_1 \to R_2 \\ R_3 + (-2)R_1 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 4 & 2 \end{array}\right]$$

$\Phi = 2$, subroutine A: entry in [row 2, col 2] = –2, multiply $-\frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 4 & 2 \end{array}\right] -\tfrac{1}{2}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 4 & 2 \end{array}\right]$$

$\Phi = 2$, subroutine B: goal is to get 0s above and below diagonal entry.

$i = 1, \Phi = 2$: entry in [row 1, col 2] = 1, do row operation $R_1 + (-1)(1)R_2$:

$$\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 0 \\ \hline 1 & 0 & 1 & 2 \end{array}$$

(working on Row 1) (R_1)
(number now in row 1, col 2) ((1))
(diagonal row 2, col 2 under consideration) (R_2)

$i = 3, \Phi = 2$: entry in [row 3, col 2] = –1, do row operation $R_3 + (-1)(-1)R_2$:

$$\begin{array}{ccc:c} 0 & -1 & 4 & 2 \\ 0 & 1 & -2 & 0 \\ \hline 0 & 0 & 2 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 4 & 2 \end{array}\right] \begin{array}{l} R_1 + (-1)R_2 \to R_1 \\ \\ R_3 + (1)R_2 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$\Phi = 3$, subroutine A: entry in [row 3, col 3] = 2, multiply $\frac{1}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 4 & 2 \end{array}\right] \tfrac{1}{2}R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\Phi = 3$, subroutine B: goal is to get 0s above and below diagonal entry.

$i = 1, \Phi = 3$: entry in [row 1, col 3] = 1, do row operation $R_1 + (-1)(1)R_3$:

$$\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ \hline 1 & 0 & 0 & 1 \end{array}$$

(working on Row 1) (R_1)
(number now in row 1, col 3) ((1))
(diagonal row 3, col 3 under consideration) (R_3)

$i = 2$, $\Phi = 3$: entry in [row 2, col 3] = –2, do row operation $R_2 + (-1)(-2)R_3$:

$$\begin{array}{cccc|c} & 0 & 1 & -2 & 0 \\ & 0 & 0 & 2 & 2 \\ \hline & 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 + (-1)R_3 \to R_1 \\ R_2 + (2)R_3 \to R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The last column contains our answer: $x = 1$, $y = 2$, $z = 1$.
Written in coordinate form our answer is $(x, y, z) = (1, 2, 1)$.

Note that our answer checks out.

$$\begin{cases} 3(1) + 2 + 1 = 6 \\ 1 + 2 - 1 = 2 \\ 2(1) + 2 + 2(1) = 6 \end{cases} \checkmark$$

A final matrix that looks like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right]$$

means there is no solution.
See Lecture Example B
and text Example 3.

A final matrix that looks like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Means that there are infinite solutions
and that parameter(s) are needed.
See Lecture Example C
and text Example 4 and Example 5..