

## Calculus 131, section 10.1 Solutions of Linear Systems

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Example A: Consider the system of three equations with three variables:

$$\begin{aligned}3x + y + z &= 6 \\x + y - z &= 2 \\2x + y + 2z &= 6\end{aligned}$$

Solving with a series of additions and eliminations might look like this:

First do two pairs of addition/eliminations to eliminate the  $z$ :

$$\begin{cases} 3x + y + z = 6 \\ x + y - z = 2 \end{cases} \Rightarrow 4x + 2y = 8$$

$$\begin{cases} x + y - z = 2 \\ 2x + y + 2z = 6 \end{cases} \Rightarrow \begin{cases} 2x + 2y - 2z = 4 \\ 2x + y + 2z = 6 \end{cases} \Rightarrow 4x + 3y = 10$$

Next, pair these two equations to eliminate the  $x$ :

$$\begin{cases} 4x + 2y = 8 \\ 4x + 3y = 10 \end{cases} \Rightarrow \begin{cases} -4x - 2y = -8 \\ 4x + 3y = 10 \end{cases} \Rightarrow y = 2$$

Then, do a back-substitution in one of the intermediate equations to solve for  $x$ :

$$4x + 2y = 8 \Rightarrow 4x + 2(2) = 8 \Rightarrow 4x = 4 \Rightarrow x = 1$$

Follow this with a back-substitution in one of the original equations to solve for  $z$ :

$$3x + y + z = 6 \Rightarrow 3(1) + (2) + z = 6 \Rightarrow z = 1$$

Finally, state the answer in coordinate form:  $(x, y, z) = (1, 2, 1)$ .

Example A has easy numbers and only three variables. With more variables, more equations, and coefficients that are not all 1s, 2s and 3s, the addition/elimination process can get cumbersome and confusing. Enter the Gauss-Jordan method.

$$\begin{cases} 3x + y + z = 6 \\ x + y - z = 2 \\ 2x + y + 2z = 6 \end{cases} \text{ is written as a } 3 \times 4 \text{ augmented matrix } \left[ \begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 2 & 6 \end{array} \right].$$

In the Math 131 text, the description of the Gauss-Jordan method tries to get upper and lower triangles with entries equaling 0 first, then multiplying rows as needed to get diagonal entries equaling 1. The algorithm described below accomplishes these goals in the opposite order.

For an augmented matrix representing a system of  $n$  equations in  $n$  variables:

Identify the diagonal, cells [row 1, col 1], [row 2, col 2], ..., [row  $n$ , col  $n$ ]. If any diagonal entry = 0, interchange rows so that each diagonal entry  $\neq 0$ .

Goals: A) in diagonal, all entries = 1, B) in triangles both above and below the diagonal, all entries = 0.

Note: To make the process flow a little more smoothly, you may want to multiply rows by constants to clear decimals and fractions, and (if possible) interchange rows so that entry in [row 1, col 1] = 1.

**Gauss-Jordan algorithm:** Repeat for each column  $\Phi = 1, 2, \dots, n$ .

**subroutine A:** If needed, multiply row  $\Phi$  by reciprocal of entry in [row  $\Phi$ , col  $\Phi$ ]. (Goal A)

**subroutine B:** For all rows  $R_i$  above and below row  $\Phi$  where [row  $i$ , col  $\Phi$ ]  $\neq 0$ ,  
do  $R_i + (-1)$  [entry in row  $i$ , col  $\Phi$ ] times  $R_\Phi$ . (Goal B)

Example A from above, worked out in step-by-step detail using this algorithm, can be downloaded by linking to “10.1 extra [Gauss-Jordan algorithm](#) (provided by Tim Pilachowski)” on my Math 131 page. I strongly recommend that you do the Gauss-Jordan method for yourself first, as practice, then use the handout to check yourself.

Example B: You have probably run across an equation in one variable, or a system of equations in two variables, where there is no solution. The equations would be two lines which are parallel, thus never intersect. Here’s an easy example of three equations in three variables with no solution.

$$\begin{cases} 2x + y + z = 4 \\ -x + y - z = 1 \\ 2x - 2y + 2z = -1 \end{cases}$$

Example C: You have probably seen an equation in one variable which is an identity, true for all values of  $x$ . In a system of equations in two variables, if the two equations would graph the same line, there are an infinite number of solutions: all of the points on that line. Here’s how we’ll recognize and handle this scenario in multiple equations and variables. *answer:*  $(w, x, y, z) = (7 - 6z, 3 - 2z, 9z - 8, z)$

**Important note:** In my augmented matrix, my columns are the coefficients of  $w$ ,  $x$ ,  $y$ , and  $z$  in that order. Your text’s example has columns  $x$ ,  $y$ ,  $z$ , then  $w$ . Neither version is “more correct” than the other. Just make sure you keep track of how your matrix is set up.

$$\begin{cases} w + x + y - z = 2 \\ 2w - x + y + z = 3 \\ w + 2x + y + z = 5 \\ -w + 6x + y - 3z = 3 \end{cases}$$

Example C continued: You'll probably need the space below.

In Example 5, the text solves a system of three equations in four variables, a scenario which would always have infinite solutions and need parameter(s). In the case of Example 5, two parameters ( $z$  and  $w$ ) are needed.

Example D: Be careful that, when you set up your augmented matrix, each column represents exactly one variable.

$$\begin{cases} 3y + z = -2 \\ 2x - y = 3 \\ x + z = 2 \end{cases} \Rightarrow \begin{cases} 0x + 3y + 1z = -2 \\ 2x - 1y + 0z = 3 \\ 1x + 0y + 1z = 2 \end{cases} \text{ is written as a } 3 \times 4 \text{ augmented matrix } \left[ \begin{array}{ccc|c} 0 & 3 & 1 & -2 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 1 & 2 \end{array} \right].$$

Preliminary check: Since there is a zero in diagonal cell [row1, col 1] we'll need to swap rows. In this case, if we exchange  $R_1$  with  $R_3$ , we get the added bonus that diagonal element [row 1, col 1] in the new matrix equals 1. After the row exchange has been done, the first step  $\Phi = 1$  of Gauss-Jordan is already mostly finished, since [row 1, col 1] = 1 and [row 3, col 1] = 0. I'll leave it to you to use the Gauss-Jordan method to show that the solution is  $(x, y, z) = (1, -1, 1)$ .