Calculus 131, section 10.4 Matrix Inverses

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Vocabulary from 10.2: A matrix with the same number of rows as columns is a **square matrix**, for example

$$B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 2 & -3 & 2 \end{bmatrix} \text{ is a } 3 \times 3 \text{ square matrix and } C = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \text{ is a } 2 \times 2 \text{ square matrix.}$$

Now a new definition: The **identity matrix** I_n is a square matrix such that, for any $n \times n$ square matrix A, it will be true that $AI = IA = A$.

An identity matrix will necessarily be a square matrix. Why? Recall NP vs. PN from Lecture 10.3. For non-square matrices, AI would be a different size than IA, so they could not be equal.

Our goal now is to find, if possible, a **multiplicative inverse** matrix A^{-1} such that, for an $n \times n$ square matrix A, it will be true that $AA^{-1} = A^{-1}A = I_n$. Note: Not all square matrices have an inverse!

An identity matrix I_n will necessarily have diagonal entries = 1 and upper and lower triangle entries = 0. (Sound familiar?)

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example A:

Given $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 2 & -3 & 2 \end{bmatrix}$, find B^{-1} (if possible). *answer*: $B^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

I'll leave it to you to check that check that $BB^{-1} = I_3$, as practice in matrix times matrix multiplication.

Example A continued: (You'll probably need the space below.)

Example B: Given $C = \begin{bmatrix} 6 & 2 \\ 12 & 4 \end{bmatrix}$, find C^{-1} (if possible). *answer*: C has no inverse.

Your text derives a formula for the inverse of a 2 × 2 square matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

I usually don't recommend memorizing this, because it can go wrong so very easily.
Note, however, that it does tell us which 2 × 2 matrices don't have an inverse. When $ad - bc = 0$, the matrix does not have an inverse. The formula $ad - bc$ is called the **determinant** of the 2 × 2 matrix A.

Theory for solving matrix equations: $AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$.

(See your text for the detailed explanation.) Note that it *must* be $X = A^{-1}B$. Don't do $B\,A^{-1}$, which is not the same thing at all: matrix times matrix multiplication is not commutative!

Example C: Solve the system of equations $\begin{cases} 2x + 3y = 5 \\ 3x - y = -5 \end{cases}$ using the inverse of a matrix.

answer: $x = -\frac{10}{11}, \;\; y = \frac{25}{11}$