Calculus 131, section 10.5 Eigenvalues and Eigenvectors

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So far we've had Gauss-Jordan (10.1), addition and subtraction of matrices (10.2), scalar times matrix multiplication and matrix times matrix multiplication (10.3), and inverse of a matrix (10.4). Now we get a new way to work with matrices: eigenvalues and eigenvectors.

As a preliminary, we need to calculate the **determinant** of a square matrix. Loosely defined, the determinant of a square matrix is "the products of the negative-slope diagonals minus the products of the positive-slope diagonals". Examples A and B below will demonstrate the process.

Example A:

Given $C = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$, find det($C$), the determinant of the matrix C. *answer*: -5.

For a 2×2 square matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = -bc + ad = ad - bc$.

The formula given in section 10.4 for the inverse of a 2×2 square matrix can be written thusly:

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $\det(A) = 0$, the matrix A does not have an inverse.

Example B:

Given $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 2 & -3 & 2 \end{bmatrix}$, find det($B$). *answer*: 3

Moving on to eigenvalues: For a square matrix M, the number λ is an **eigenvalue** if and only if there exists a non-zero vector X such that $MX = \lambda X$. That is, the matrix times matrix multiplication MX is a scalar multiple of the matrix X. The matrix X is a particular type called an **eigenvector.** Your text points out that the word "eigen" is German, meaning characteristic.

Theory: The matrix O is the zero matrix, i.e. a matrix in which all entries = 0.

$$MX = \lambda X \quad \Rightarrow \quad MX = \lambda IX \quad \Rightarrow \quad MX - \lambda IX = O \quad \Rightarrow \quad (M - \lambda I)X = O$$

In a course in applied linear algebra, it is proven that this is mathematically equivalent to solving

$$\det(M - \lambda I) = 0.$$

Example C:

Given $D = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, find the eigenvalues and a corresponding eigenvector for each one.

answers: $-2, 1, 3$; $\lambda = -2 \;\Rightarrow\; X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\lambda = 1 \;\Rightarrow\; X = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$, $\lambda = 3 \;\Rightarrow\; X = \begin{bmatrix} 2 \\ -5 \\ 10 \end{bmatrix}$

Theory: Eigenvalues and the corresponding eigenvectors can be used to describe populations, particularly those which grow or shrink proportionately.

no. juveniles future = proportion of juveniles' offspring + proportion of adults' offspring

$$x_1(t+1) = px_1(t) + qx_2(t)$$
$$x_2(t+1) = sx_1(t) + rx_2(t)$$

no. adults future = proportion surviving juveniles + proportion of surviving adults

(In your text's example at the beginning of 10.5, $rx_2(t) = 0$ since none of the adults survive beyond two years.)

The matrix $M = \begin{bmatrix} p & q \\ s & r \end{bmatrix}$ is called a Leslie matrix after the first person to write about them. Your text gives the reference information in a footnote.

Example D: Given the Leslie matrix $M = \begin{bmatrix} 0.1 & 0.5 \\ 0.4 & 0.9 \end{bmatrix}$, a) find a population of size 12,000 for which the proportion of the population in each age group stays the same from one year to the next, and b) tell by what factor the population grows or declines each year. *answers*: $x_1 = 4000$ $x_2 = 8000$ factor = 1.1

How should we interpret eigenvalues in this type of situation?

Our factor of growth/decline was the eigenvalue $\lambda = 1.1$.

It means that the populations (juveniles and adults) are
- increasing to 1.1 times their previous size
- increasing to 110% of their previous size
- increasing by 10% over their previous size

If the factor of growth/decline had been equal to 1 (eigenvalue *lambda* = 1), it would indicate the population is at an equilibrium, i.e. is staying at the same numbers over time.

If the factor of growth/decline had been less than 1, it would indicate the population is declining, decreasing in size.

For example, an eigenvalue *lambda* = 0.8 means that the populations (juveniles and adults) are
- shrinking to 0.8 times their previous size
- shrinking to 80% of their previous size
- shrinking by 20% of their previous size

Homework exercise #18 from section 10.5 is an example of a Leslie matrix for which both eigenvalues are workable, i.e. neither is extraneous. Work this one through for practice. You should get

$\lambda = 0.8$ with solutions $x_1 = \dfrac{20000}{3}$ and $x_2 = \dfrac{10000}{3}$ and

$\lambda = 0.3$ with solutions $x_1 = \dfrac{10000}{3}$ and $x_2 = \dfrac{20000}{3}$.

In both scenarios, the population is declining, with the latter being a faster decline than the former.