Calculus 131, section 11.1 Elementary & Separable Differential Equations

notes prepared by Tim Pilachowski

You know how to differentiate a number of types of equations:

$$f(x) = ax^5 \Rightarrow f'(x) = 5ax^4 \qquad g(x) = e^{\sin t} \Rightarrow \frac{dx}{dt} = \cos t\, e^{\sin t} \Rightarrow \frac{d^2x}{dt^2} = e^{\sin t}\left(-\sin t + \cos^2 t\right)$$

In some cases we can (and did) integrate to move from a derivative (i.e. rate of change) back to the original function (i.e. an amount).

$y' = \cos(t) \quad \Rightarrow$ via an antiderivative: $y = \int \cos(t)\, dt = \sin(t) + C$

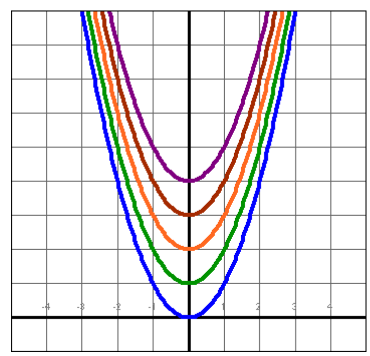
$y' = 2x\left(x^2+3\right)^7 \quad \Rightarrow$ via substitution: $y = \int 2x\left(x^2+3\right)^7\, dx = \frac{1}{8}\left(x^2+3\right)^8 + C$

$y' = x^2 \ln x \quad \Rightarrow$ via integration by parts: $y = \int x^2 \ln x\, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

Now, however, the process of solving a differential equation (DE) is going to be more involved.

Terminology note: The *order* of a differential equation is equal to the highest-order derivative involved. So $\frac{dy}{dt} = 2y - t$ is a first-order DE, and $\frac{d^2y}{dt^2} = 3\frac{dy}{dt} + 2y - t$ is a second-order DE.

Usually a DE will have many solution functions. Think about the integration we've done up to this point: You had to remember to write "+ *C*" as part of your answer. Each possible value for the constant *C* (which in some applications is also called a parameter) represents a different function.



For example, $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 + 3$, $y = x^2 + 4$, etc, are all solutions of the DE $y' = 2x$.

Note that the placement of the constants in your answers to a DE will not always be "+ *C*" ! (See Example A below.)

Solutions that look like $y = x^2 + C$ are called ***general*** solutions, because one can derive all solutions by substituting an appropriate value for the constant *C* (also sometimes called a parameter) in a given situation. A solution that looks like $y = x^2 + 3$ is called a ***particular*** solution because there are no parameters which will change value in varied situations. An initial condition will provide a point of reference, allowing us to apply the general solution to a particular instance. (A cautionary note: While it is sometimes tempting to do so, "initial condition" should not be understood to always mean "when *time* = 0".)

Example A: Find a function *f*(*x*) such that $x^2 - \frac{dy}{dx} = 2x - 3$ and $y = 3$ when $x = 2$.

answer: $y = \frac{1}{3}x^3 - x^2 + 3x - \frac{5}{3}$

While many of the differential equations you encounter in this class will seem to be somewhat capricious, in the real world differential equations have many useful applications. Observations about how something changes (grows or declines, increases or decreases) can lead researchers to an equation which describes its behavior.

We were able to rewrite the equation in Example A so that we could do a simple integration. Others aren't as easy. One type of DE, a *separable* differential equation, can be rewritten to make it almost as simple. The method will be to separate elements containing the x variable from ones containing the y variable.

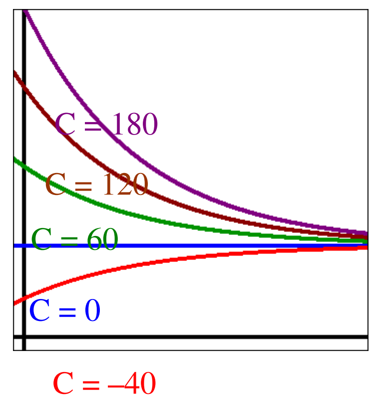
Separable DEs will generally have one of two forms:

$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \frac{dy}{g(y)} = f(x)\,dx \qquad\qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \Rightarrow \quad g(y)\,dy = f(x)\,dx$$

That is, we'll be able to use multiplication and division to put all of the y-stuff on one side and all the x-stuff or t-stuff on the other.

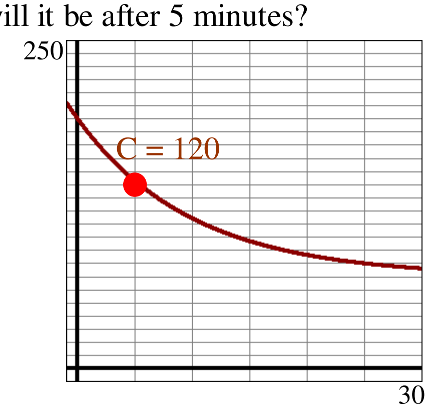
Example B: The temperature y of a cup of coffee changes with time (Newton's Law of Cooling). In a room that is 70 degrees Farenheit: $\frac{dy}{dt} = -0.1(y-70)$. Find all solutions y.

answer: $y = Ce^{-0.1t} + 70$

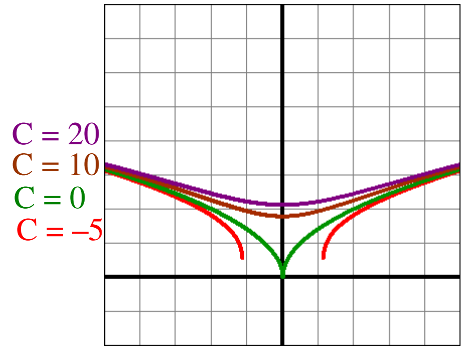


Example B extended: If the cup of coffee began at 190° F, what temperature will it be after 5 minutes?

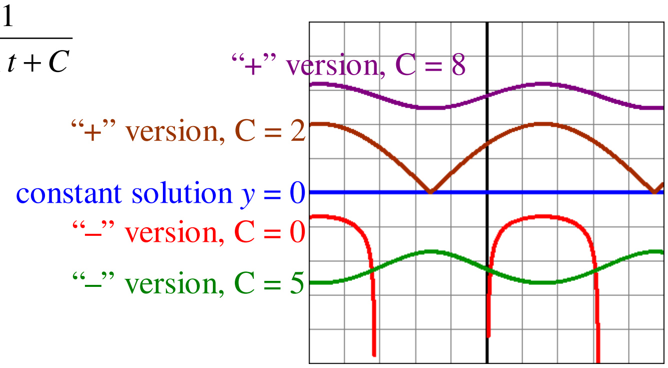
answer: $120e^{-0.5} + 70$ degrees (exact), $\approx 142.78°$



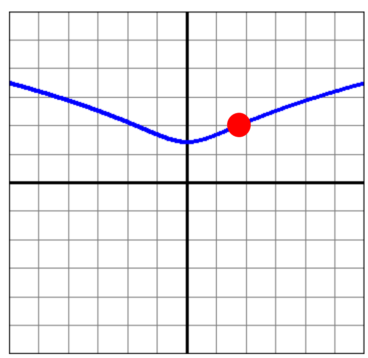
Example C: Find all solutions of the DE $\frac{dy}{dt} = \frac{2t}{y^3}$. *answer*: $y = \sqrt[4]{4t^2 + C}$



Example D: Solve $\frac{dy}{dt} + y^3 \cos t = 0$. *answer*: $y = \pm\sqrt{\frac{1}{2\sin t + C}}$



Example E: Solve $\sqrt{t^2+1}\,\frac{dy}{dt} = \frac{t}{y}$, $y\left(\sqrt{3}\right) = 2$. *answer*: $y = \sqrt{2\sqrt{t^2+1}}$



A separable DE might also require "integration by parts". We don't have time to work through an example in Lecture. Review that on your own.

Example F: Solve $x\frac{dy}{dx} = x^2 + y^2$

answer: This DE is not separable!! The addition makes separation of variables impossible. We'd need either a product or quotient of $f(x)$ and $g(y)$. [See the beginning of this Lecture.]

Example G: One injection of 10 mg of a medication is given to a patient. The amount of the drug in the bloodstream decreases by 8% per hour. a) Set up and solve a differential equation that is satisfied by $P(t)$, the amount of the medication in the bloodstream of the patient. b) Estimate the amount of the drug in the bloodstream after 24 hours (to the nearest tenth). *answers*: $P = 10e^{-0.08t}$, ≈ 1.5 mg

Example H: A lack of available resources limits a population of sheldons to 1000. With an initial population of 50, and a growth constant of 2%, write a differential equation and then find a particular solution to describe this population's growth. *answer*: $y = 1000 - 950e^{-0.02x}$

We can now use this model to predict growth of the sheldon population, including an estimate of how long it will take to reach its maximum sustainable number.

Note that as time goes on ($\lim_{x \to \infty}$), the population will approach its maximum asymptotically. That is, on the graph, the horizontal line $y = 1000$ is an asymptote.

Note, too, that the differential equation and solution process here was very much like the Newton's Law of Cooling example earlier. Other examples would be spread of an illness from contact with a source (e.g. food poisoning from contaminated eggs) or information spread by poster or newsletter.