Calculus 131, section 11.2 1st Order Linear Differential Equations

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Recall the clue telling you that you have a separable differential equation (DE): Separable DEs will generally have one of two forms.

$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int \frac{1}{g(y)}\,dy = \int f(x)\,dx \qquad\qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx$$

When a DE is not separable, we'll need another method. Today we'll develop one for linear first-order differential equations. These are of the form $\frac{dy}{dx} + P(x) * y = Q(x)$. "First-order" means y' not y'', y''', etc. "Linear" means y, not y^2, y^3, etc.

Before we get back to this model, we need some preliminaries.

Example A: Solve $(\cos t)y' - (\sin t)y = t + 20$. *answer*: $y = \frac{t^2}{2\cos t} + \frac{20t}{\cos t} + \frac{C}{\cos t} = \frac{t^2 + 40t + C}{2\cos t}$

Note the position of "+C" in this and other answers!

Example B: Solve $e^{10t}y' + 10e^{10t}y = t + 20$. *answer*: $y = e^{-10t}\left(\frac{1}{2}t^2 + 20t + C\right)$

This example will be particularly helpful in learning the process for solving linear first-order DEs.

Theory: Consider the DE $t\,y' = t^2 + 3y, \quad t > 0$. As it is, we cannot rearrange to look like $f\,y' + f'\,y = \ldots$ The coefficient of y is not the derivative of the coefficient of y'. However, all is not lost. Recall the general form first-order linear DE: $\frac{dy}{dx} + P(x) * y = Q(x)$. To solve this type of DE we'll do some clever things. First we'll find $\int P(x)\,dx$, i.e. an anti-derivative of $P(x)$, the coefficient of y in the first order linear DE.

Next, we'll form the integrating factor $e^{\int P(x)\,dx}$ and multiply both sides of the DE as follows.

$$\left(e^{\int P(x)\,dx}\right)y' + \left(e^{\int P(x)\,dx}\right)a(t)\,y = \left(e^{\int P(x)\,dx}\right)Q(x)$$
$$f \quad g' + \quad f' \quad g =$$
$$\frac{d}{dx}[f * g] =$$
$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\, y\right] = \left(e^{\int P(x)\,dx}\right)Q(x)$$
$$\int \frac{d}{dt}\left[e^{\int P(x)\,dx}\, y\right]dx = \int\left(e^{\int P(x)\,dx}\right)Q(x)\,dx$$
$$e^{\int P(x)\,dx}\, y = \int\left(e^{\int P(x)\,dx}\right)Q(x)\,dx$$

The right-hand side may simplify nicely so we can integrate, or we can use substitution or integration by parts.

Example C: Solve $x\frac{dy}{dx} = x^2 + 3y, \quad x > 0$. *answer*: $y = -x^2 + Cx^3, \quad x > 0$

1) Rewrite in standard form.

2a) Find $\int P(x)\,dx$.

2b) Find the integrating factor.

3) Multiply by the integrating factor and integrate both sides.

Example C extended: Given the initial condition $y(1) = 2$, find the particular solution. *answer*: $y = 3x^3 - x^2$

Note that since $\int P(x)\,dx$ can be *any* anti-derivative, we might as well pick the simplest one.

In using this process, it will be important to correctly identify $P(x)$, and also to correctly construct the *integrating factor* $I(x) = e^{\int P(x)\,dx}$. It is called an integrating factor because it transforms the left-hand side into a form that can be integrated because it is the result of differentiating by the Product Rule.

Example D: Solve $\frac{dy}{dx}+2xy=6x^3$. *answer*: $y=3x^2-3+Ce^{-x^2}$

Examples E: $x\frac{dy}{dt}=x^2+y^2$ is not linear (because the y^2), and $x\frac{d^2y}{dx^2}=x^2+y$ is not first-order (because it has $\frac{d^2y}{dx^2}$)!

Example F: An anti-coagulant is introduced intravenously at a rate of 0.5 mg per hour. On a continuous basis 2% of the drug in the bloodstream is absorbed into the body. a) Set up and solve a differential equation that is satisfied by $y(t)$, the amount of anti-coagulant the bloodstream of the patient. b) Determine the equilibrium amount of the anti-coagulant in the bloodstream of the patient. *answer*: $y=25+Ce^{-0.02t}$, 25 mg