Calculus 131, section 11.3 Euler's Method

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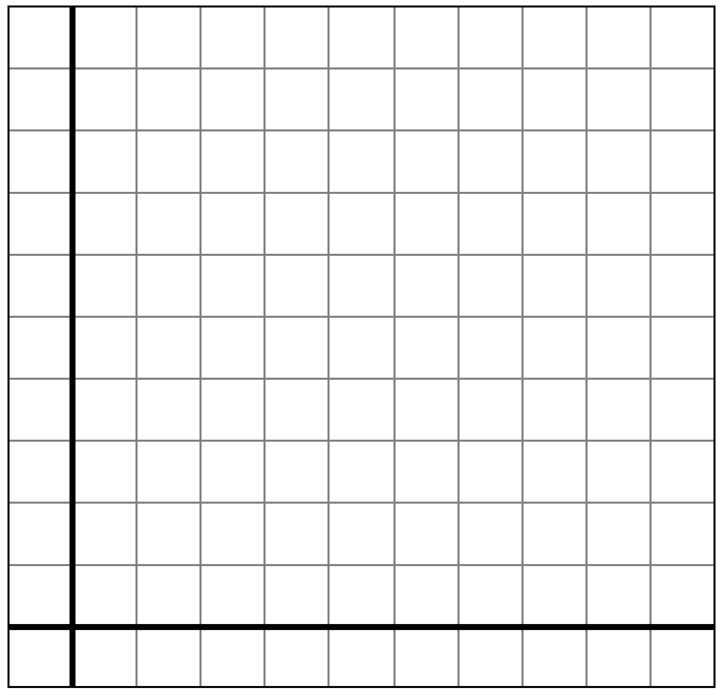
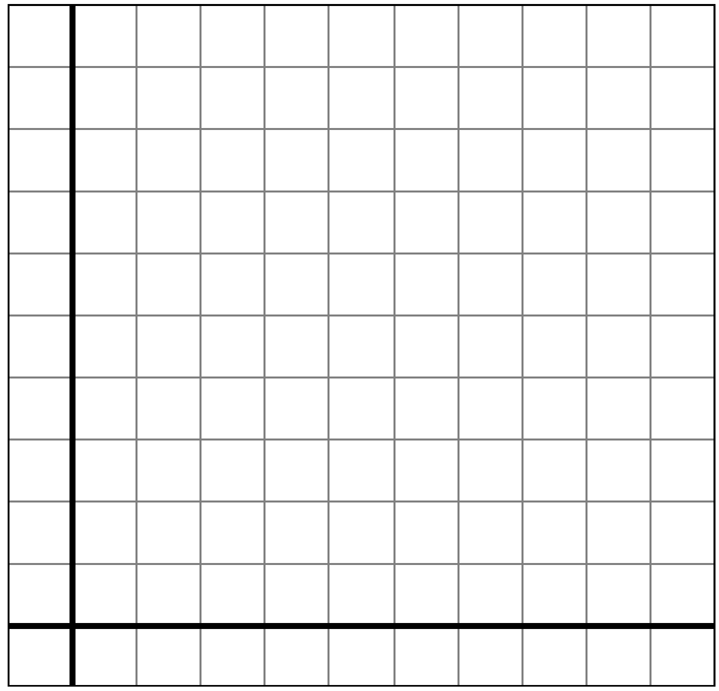
In sections 11.1 and 11.2, we solved separable and first-order liner differential equations. But what can we do when the DE doesn't fall into either of these categories? That is, can we find a solution for a differential equation $\frac{dy}{dx} = g(x, y)$? The answer is "Yes, with conditions" and we'll do so by sketching a polygonal approximation using the information at hand. We'll begin with a specific initial value and follow a particular path—one that values of $\frac{dy}{dx}$ lead us to.

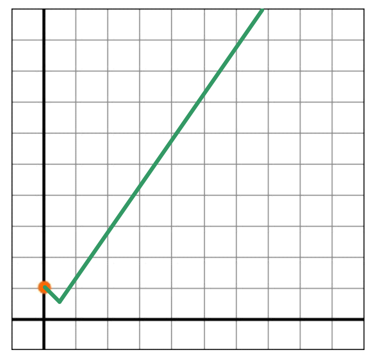
Example A: Given a differential $y' = 3t - 2y + 1$ and an initial value $y(0) = 1$, sketch an approximate solution y.



polygonal approximation when $h = 1$

Example A extended: Given a differential $y' = 3t - 2y + 1$ and an initial value $y(0) = 1$, sketch an approximate solution y, this time using $h = \Delta t = ½ = 0.5$.





polygonal approximation when $h = ½$
(actual graph pictured in Example A above)

The process used in Example A is the basis for Euler's Method, named after Leonhard Euler. We begin with a differential equation $y' = g(x, y)$ and an interval $a \le x \le b$. The interval is divided up into n sub-intervals, each of width $\Delta t = x = \frac{b-a}{n}$. Initial values $x_0 = a$ and y_0 are given. Then comes a series of calculations.

$$x_i = x_{i-1} + h \qquad y_{i+1} = y_i + g(x_i, y_i) * h$$

Example B: Given $y' = -x^2(y-1)$ and $y(0) = 2$, use Euler's method with $n = 6$ to estimate $y(3)$.

	$y_3 = \frac{15}{8} - \frac{7}{8} * \frac{1}{2} = \frac{23}{16}$	$y'\left(\frac{3}{2}, \frac{23}{8}\right) = -\left(\frac{3}{2}\right)^2 \left(\frac{23}{16} - 1\right) = -\frac{63}{64}$
	$y_4 = \frac{23}{16} - \frac{63}{64} * \frac{1}{2} = \frac{121}{128}$	$y'\left(2, \frac{121}{128}\right) = -(2)^2 \left(\frac{121}{128} - 1\right) = \frac{7}{32}$
	$y_5 = \frac{121}{128} + \frac{7}{32} * \frac{1}{2} = \frac{135}{128}$	$y'\left(\frac{5}{2}, \frac{135}{128}\right) = -\left(\frac{5}{2}\right)^2 \left(\frac{135}{128} - 1\right) = -\frac{175}{512}$
	$y_6 = \frac{135}{128} - \frac{175}{512} * \frac{1}{2} = \frac{905}{1024}$	(not needed)

answer: $y(3) \approx y_6 = \frac{905}{1024} = 0.8837890625$

Note 1: If you solve this one for practice, separation of variables is the most efficient method. You should get $y = 1 + e^{-x^3/3}$. When $x = 3$, $y = 1 + e^{-9} \approx 1.00012341$.

Note 2: Euler's method is an ***approximation*** process which has its own inherent error. I used fractions to minimize error due to rounding. In actual practice, decimals to a prescribed accuracy would be used. (You could set up a spreadsheet.)

Note 3: As the number of sub-intervals, n, becomes large, the compounded errors also increase. If we take Example B and increase the number of intervals to 12, we get an approximate value of 1, which is the value of the horizontal asymptote. The lesson: The value of n can be too small, or too large, so we shoot for "just right".

x	y estimated	y' calculation
0	2	0
0.25	2	-0.0625
0.5	1.984375	-0.24609375
0.75	1.922851563	-0.519104004
1	1.793075562	-0.793075562
1.25	1.594806671	-0.929385424
1.5	1.362460315	-0.815535709
1.75	1.158576388	-0.485640188
2	1.037166341	-0.148665364
2.25	1	0
2.5	1	0
2.75	1	0
3	1	0