Systems of Differential Equations - Better than the Book, I Hope

1. **The Situation:** We know that if a population grows by 40% per year that if x_1 is the population then $\frac{dx_1}{dt} = 0.4x_1$. This is because the change (derivative) is 40% of the original.

   So suppose there are two populations, x_1 and x_2. Suppose each year x_1 grows by 10% of it's own population but suppose it also grows by 120% of x_2's population - perhaps x_2 grows a lot and most move to x_1. Then we'd have $\frac{dx_1}{dt} = 0.1x_1 + 1.2x_2$. Suppose in addition x_2 grows by 40% of x_1's population and by 30% of its own. Then we'd have $\frac{dx_2}{dt} = 0.4x_1 + 0.3x_2$.

   Together we call this a *system of differential equations*:

$$\frac{dx_1}{dt} = 0.1x_1 + 1.2x_2$$
$$\frac{dx_2}{dt} = 0.4x_1 + 0.3x_2$$

   A solution is a pair of functions $x_1 = ...$ and $x_2 = ...$, both functions of time t, which satisfy both. How on earth would we find this?

2. **Harder Still:** What if both populations are additionally affected by some outside influence. For example perhaps x_1 gains e^t new people each year from some third source and x_2 gains 7 new people each year from some third source. Then we have

$$\frac{dx_1}{dt} = 0.1x_1 + 1.2x_2 + e^t$$
$$\frac{dx_2}{dt} = 0.4x_1 + 0.3x_2 + 7$$

   How would we solve this?! We'll see!

3. **Theorem:** There is a basic premise underlying this section which we will mention but not prove. The proof requires some serious linear algebra.

   Suppose M is a 2×2 matrix. Suppose λ_1 is an eigenvalue with some eigenvector $\begin{bmatrix} a \\ b \end{bmatrix}$ and λ_2 is an eigenvalue with some eigenvector $\begin{bmatrix} c \\ d \end{bmatrix}$. Then it turns out that:

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1}$$

   We'll write

$$M = PDP^{-1} \quad \text{with } P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

4. **Example of Theorem:** For example if we start with

$$M = \begin{bmatrix} 0.1 & 1.2 \\ 0.4 & 0.3 \end{bmatrix}$$

   then we find the eigenvalues and eigenvectors we get (work omitted) $\lambda_1 = 0.9$ with $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\lambda_2 = -0.5$ with $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus we can check that the following is in fact true - check it:

$$\begin{bmatrix} 0.1 & 1.2 \\ 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.9 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}^{-1}$$

5. **That's Crazy Talk! Now What?** Suppose we have a system of differential equations of the form

$$\begin{aligned}\frac{dx_1}{dt} &= m_{11}x_1 + m_{12}x_2 + q_1(t)\\ \frac{dx_2}{dt} &= m_{21}x_1 + m_{22}x_2 + q_2(t)\end{aligned}$$

We can rewrite this with matrices and vectors as

$$\begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{bmatrix} = \begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

Then we can define $X = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $M = \begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}$ and $Q = \begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$ and then rewrite this system as

$$\frac{dX}{dt} = MX + Q$$

Now let's play a bit. Define $Y = P^{-1}X$ so that $X = PY$. Since P is all constants we have $\frac{dX}{dt} = P\frac{dY}{dt}$. Then we work with the equation above:

$$\begin{aligned}\frac{dX}{dt} &= MX + Q\\ P\frac{dY}{dt} &= PDP^{-1}X + Q\\ \frac{dY}{dt} &= DP^{-1}X + P^{-1}Q\\ \frac{dY}{dt} &= DY + P^{-1}Q\end{aligned}$$

In theory if we can solve $\frac{dY}{dt} = DY + P^{-1}Q$ then we can find $X = PY$. In practice because D is diagonal it turns out that solving this is often not hard at all (famous last words.)

6. **May I See an Example?** Consider for example:

$$\frac{dx_1}{dt} = 0.1x_1 + 1.2x_2 + e^t$$
$$\frac{dx_2}{dt} = 0.4x_1 + 0.3x_2 + 7$$

We have $M = \begin{bmatrix} 0.1 & 1.2 \\ 0.4 & 0.3 \end{bmatrix}$.

The eigenstuff gives us $D = \begin{bmatrix} 0.9 & 0 \\ 0 & -0.5 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$ so $P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$.

We have $Q = \begin{bmatrix} e^t \\ 7 \end{bmatrix}$.

We will therefore solve

$$\frac{dY}{dt} = DY + P^{-1}Q$$
$$\frac{dY}{dt} = \begin{bmatrix} 0.9 & 0 \\ 0 & -0.5 \end{bmatrix} Y + \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} e^t \\ 7 \end{bmatrix}$$
$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{7}e^t + 2 \\ -\frac{2}{7}e^t + 3 \end{bmatrix}$$
$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 0.9y_1 \\ -0.5y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{7}e^t + 2 \\ -\frac{2}{7}e^t + 3 \end{bmatrix}$$
$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 0.9y_1 + \frac{1}{7}e^t + 2 \\ -0.5y_2 - \frac{2}{7}e^t + 3 \end{bmatrix}$$

which corresponds to the system

$$\frac{dy_1}{dt} = 0.9y_1 + \frac{1}{7}e^t + 2$$
$$\frac{dy_2}{dt} = -0.5y_2 - \frac{2}{7}e^t + 3$$

These are actually first-order linear if we view them as:

$$\frac{dy_1}{dt} - 0.9y_1 = \frac{1}{7}e^t + 2$$
$$\frac{dy_2}{dt} + 0.5y_2 = -\frac{2}{7}e^t + 3$$

Remember: First order linear means of the following form where we've used t in place of x:

$$\frac{dy}{dt} + P(t)y = Q(t)$$

The first has $s(t) = -0.9t$ and hence solution

$$\begin{aligned}
y_1 &= e^{0.9t} \int \left(\frac{1}{7} e^t + 2 \right) e^{-0.9t} \ dt \\
&= e^{0.9t} \int \frac{1}{7} e^{0.1t} + 2e^{-0.9t} \ dt \\
&= e^{0.9t} \left[\frac{10}{7} e^{0.1t} - \frac{20}{9} e^{-0.9t} + C_1 \right] \\
&= \frac{10}{7} e^t - \frac{20}{9} + C_1 e^{0.9t}
\end{aligned}$$

The second has $s(t) = 0.5t$ and hence solution

$$\begin{aligned}
y_2 &= e^{-0.5t} \int \left(-\frac{2}{7} e^t + 3 \right) e^{0.5t} \ dt \\
&= e^{-0.5t} \int \left(-\frac{2}{7} e^{1.5t} + 3e^{0.5t} \right) \ dt \\
&= e^{-0.5t} \left[-\frac{4}{21} e^{1.5t} + 6e^{0.5t} + C_2 \right] \\
&= -\frac{4}{21} e^t + 6 + C_2 e^{-0.5t}
\end{aligned}$$

So then in closing

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= X \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= PY \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{10}{7} e^t - \frac{20}{9} + C_1 e^{0.9t} \\ -\frac{4}{21} e^t + 6 + C_2 e^{-0.5t} \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 3 \left(\frac{10}{7} e^t - \frac{20}{9} + C_1 e^{0.9t} \right) - 2 \left(-\frac{4}{21} e^t + 6 + C_2 e^{-0.5t} \right) \\ 2 \left(\frac{10}{7} e^t - \frac{20}{9} + C_1 e^{0.9t} \right) + \left(-\frac{4}{21} e^t + 6 + C_2 e^{-0.5t} \right) \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{14}{3} e^t + 3C_1 e^{0.9t} - 2C_2 e^{-0.5t} - \frac{56}{3} \\ \frac{8}{3} e^t + 2C_1 e^{0.9t} + C_2 e^{-0.5t} + \frac{14}{9} \end{bmatrix}
\end{aligned}$$

So when all is said and done:

$$\begin{aligned}
x_1 &= \frac{14}{3} e^t + 3C_1 e^{0.9t} - 2C_2 e^{-0.5t} - \frac{56}{3} \\
x_2 &= \frac{8}{3} e^t + 2C_1 e^{0.9t} + C_2 e^{-0.5t} + \frac{14}{9}
\end{aligned}$$

7. **Initial Values!** Suppose in addition we know that $x_1(0) = 100$ and $x_2(0) = 200$ we can plug these in and solve for both C_1 and C_2, giving us the specific solution.

$$\begin{aligned} 100 &= \frac{14}{3} + 3C_1 - 2C_2 - \frac{56}{3} \\ 200 &= \frac{8}{3} + 2C_1 + C_2 + \frac{14}{9} \end{aligned}$$

which yields

$$C_1 = \frac{650}{9} \quad \text{and} \quad C_2 = \frac{154}{3}$$

And so finally

$$\begin{aligned} x_1 &= \frac{14}{3}e^t + 3\left(\frac{650}{9}\right)e^{0.9t} - 2\left(\frac{154}{3}\right)e^{-0.5t} - \frac{56}{3} \\ x_2 &= \frac{8}{3}e^t + 2\left(\frac{650}{9}\right)e^{0.9t} + \left(\frac{154}{3}\right)e^{-0.5t} + \frac{14}{9} \end{aligned}$$

8. **Holy Macaroni!** Okay, this was pretty rough but mostly because of the Q business. Without that it's actually really easy, for example if $q_1 = q_2 = 0$ it's simple and if they're constants it's still not so bad.

9. **Bigger!** What's more, all of this is true with 3×3 matrices. What fun!

10. **Summary:** In summary the method is as follows:

    (a) Let M be the matrix which corresponds to the coefficients of x_1 and x_2 in the system.

    (b) Find the eigenvalues and eigenvectors and create the matrices D and P. Then find P^{-1}.

    (c) Solve the two equations forming $\frac{dY}{dt} = DY + P^{-1}Q$, these are (for us) separable or first-order linear.

    (d) Use $X = PY$ to find X, thereby finding x_1 and x_2.

11. Here is another easier example so you can see that this isn't always so bad!

Suppose

$$\frac{dx_1}{dt} = 0.6x_1 + 0.8x_2$$
$$\frac{dx_2}{dt} = 0.9x_1$$

We have $M = \begin{bmatrix} 0.6 & 0.8 \\ 0.9 & 0 \end{bmatrix}$.

The eigenstuff (work omitted) gives us $D = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.6 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 2 \\ 3 & -3 \end{bmatrix}$ so $P^{-1} = -\frac{1}{18}\begin{bmatrix} -3 & -2 \\ -3 & 4 \end{bmatrix}$.

We have $Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We will therefore solve

$$\begin{aligned} \frac{dY}{dt} &= DY + P^{-1}Q \\ &= \begin{bmatrix} 1.2 & 0 \\ 0 & -0.6 \end{bmatrix} Y - \frac{1}{18}\begin{bmatrix} -3 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1.2 & 0 \\ 0 & -0.6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} 1.2y_1 \\ -0.6y_2 \end{bmatrix} \end{aligned}$$

which corresponds to the system

$$\frac{dy_1}{dt} = 1.2y_1$$
$$\frac{dy_2}{dt} = -0.6y_2$$

These are separable. In fact these are of the form $\frac{dy}{dt} = ky$ and we know the solution is $y = Ce^{kt}$. Thus we have solutions

$$y_1 = C_1e^{1.2t}$$
$$y_2 = C_2e^{-0.6t}$$

So then

$$X = PY = \begin{bmatrix} 4 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} C_1e^{1.2t} \\ C_2e^{-0.6t} \end{bmatrix} = \begin{bmatrix} 4C_1e^{1.2t} + 2C_2e^{-0.6t} \\ 3C_1e^{1.2t} - 3C_2e^{-0.6t} \end{bmatrix}$$

So then the final solution is

$$x_1 = 4C_1e^{1.2t} + 2C_2e^{-0.6t}$$
$$x_2 = 3C_1e^{1.2t} - 3C_2e^{-0.6t}$$