

Calculus 131, section 11.4 Systems of Linear Differential Equations

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Example A: Foxes and rabbits share an ecological niche, and the two populations are interconnected since some of the rabbits will become food for some of the foxes. Let x_1 = size of fox population and x_2 = size of rabbit population.

(N.B. This is a *very* simplified model, and doesn't address the presence or absence of other animals. Coefficients were chosen so that the calculations involved are relatively straightforward and don't obscure the solution process.)

[rate of change of fox population] = [fox pups born] + [foxes with food source]

$$\frac{dx_1}{dt} = 0.4x_1 + 0.7x_2$$

The growth/decline of the fox population depends on both the number of foxes and the number of rabbits.

[rate of change of rabbit population] =

– rabbits which are food + rabbit kits born + vegetation food source growth

$$\frac{dx_2}{dt} = -0.3x_1 + 1.4x_2 + e^t$$

The growth/decline of the fox population depends on both the number of foxes and the number of rabbits.

We have a **system of linear differential equations**. The goal is to solve the system to get a single equation for each population that is dependent only on time t , and not on the population of the other animal.

Here's the theory and process (in general terms). We won't have time to develop the matrix algebra that justifies the process in lecture; this is a brief overview. (If you are interested, see your text and the supplement available from my Math 131 web page for a deeper explanation/justification/proof.)

Let X and Y be matrices whose elements are variable functions, let Q be a matrix whose elements are functions of t , and let M , P and D be matrices whose elements are number coefficients. For a system of two DEs in two variables, such as Example A above,

$$\begin{cases} \frac{dx_1}{dt} = \alpha x_1 + \beta x_2 + \chi(t) \\ \frac{dx_2}{dt} = \delta x_1 + \varepsilon x_2 + \phi(t) \end{cases} \Rightarrow \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \delta & \varepsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \chi(t) \\ \phi(t) \end{bmatrix} \Rightarrow [X'] = M X + Q .$$

Let P and D be matrices such that $PDP^{-1} = M$ and $X = PY$, so that $P^{-1} X = Y$ and $[X'] = P[Y']$. Then

$$\begin{aligned} [X'] &= MX + Q \\ P[Y'] &= PDP^{-1} X + Q \\ P^{-1} P[Y'] &= P^{-1} PDP^{-1} X + P^{-1} Q \\ [Y'] &= DP^{-1} X + P^{-1} Q \\ [Y'] &= DY + P^{-1} Q \end{aligned}$$

Here's the good news: A matrix D created from the eigenvalues of M and a matrix P created from the eigenvectors of M fulfills all of the necessary conditions, and if we can solve the DEs involved in $[Y'] = DY + P^{-1} Q$, then we can use back-substitution into $X = PY$ to reach our ultimate goal.

Broken down into specific steps, the process looks like this:

Step 1a) Rewrite as a matrix equation and identify M and Q .

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \delta & \varepsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \chi(t) \\ \phi(t) \end{bmatrix}$$
$$[X'] = M X + Q$$

Step 1b) Find eigenvalues and eigenvectors for matrix M (coefficients of variables x_1, x_2 , etc.)

Step 2a) Create matrix D from the eigenvalues and matrix P from the eigenvectors of Step 1.

Be careful that the columns with eigenvalues in D match the respective corresponding columns with eigenvectors in P . Side note: The interrelationships among D, M and P are what make the process work: $P^{-1}MP = D$ and $PDP^{-1} = M$.

Step 2b) Find P^{-1} .

Step 3) Set up and solve differential equations $[Y'] = D Y + P^{-1} Q$.

Step 4) To get equations for x_1 and x_2 , multiply $X = PY$.

Step 5) (if needed) Substitute initial condition values to find C_1 and C_2 .

There is a supplement on my Math 131 web page that summarizes the process and does two examples, one with first order linear DEs and the other with separable DEs. The notations are a little different, but the procedure is the same.

Example A: Foxes and rabbits share an ecological niche, and the two populations are interconnected since some of the rabbits will become food for some of the foxes. Let x_1 = size of fox population and x_2 = size of rabbit population.

$$\begin{cases} \frac{dx_1}{dt} = 0.4x_1 + 0.7x_2 \\ \frac{dx_2}{dt} = -0.3x_1 + 1.4x_2 + e^t \end{cases}$$

Step 1a) Rewrite as a matrix equation and identify M and Q . $[X'] = M X + Q$

$$M =$$

$$Q =$$

Step 1b) Find eigenvalues and eigenvectors for matrix M .

I won't have time in the lecture to actually do this step. Instead, I would encourage you to do this yourself for practice. You should get $\lambda_1 = 0.7$ with corresponding eigenvector $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ and $\lambda_2 = 1.1$ with corresponding

eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step 2a) Create matrix D from the eigenvalues and matrix P from the eigenvectors of Step 1.

$$D =$$

$$P =$$

Step 2b) Find P^{-1} .

I won't have time in the lecture to actually do this step. Instead, I would encourage you to do this yourself for practice. You should get $P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 0.25 & -0.25 \\ -0.75 & 1.75 \end{bmatrix}$.

Step 3a) Set up differential equations $[Y'] = D Y + P^{-1} Q$.

$$\frac{dy_1}{dt} =$$

$$\frac{dy_2}{dt} =$$

Step 3b) Solve the differential equations you just found.

I won't have time in the lecture to solve $\frac{dy_2}{dt} - 1.1y_2 = 1.75e^t$. Instead, I would encourage you to do this yourself for practice. You should get $y_2 = -\frac{35}{2}e^t + C_2e^{1.1t}$.

Step 4) To get equations for x_1 and x_2 , multiply $X = PY$.

Step 5) (if needed) Substitute initial condition values to find C_1 and C_2 .

Suppose I have an initial population of 20 foxes and 50 rabbits. That is, at time $t = 0$, $x_1 = 20$ and $x_2 = 50$. Find the particular solutions.

I may not have time in the lecture to do this. Instead, I would encourage you to do this yourself for practice.

You should get $C_1 = -\frac{20}{3}$, $C_2 = 90$, thus $x_1 = -\frac{70}{3}e^t - \frac{140}{3}e^{0.7t} + 90e^{1.1t}$ and $x_2 = -20e^t - 20e^{0.7t} + 90e^{1.1t}$.

TRY THE FOLLOWING TASKS YOURSELF FIRST. ONLY USE THE SOLUTIONS BELOW IF YOU NEED TO FIGURE OUT WHAT WENT WRONG.

Appendix 1. Step 1b) Find eigenvalues and eigenvectors for matrix M .

$$M - \lambda I = \begin{bmatrix} 0.4 - \lambda & 0.7 \\ -0.3 & 1.4 - \lambda \end{bmatrix}$$

$$\det(M - \lambda I) = (0.4 - \lambda)(1.4 - \lambda) - (-0.3)(0.7) = \lambda^2 - 1.8\lambda + 0.77 = (\lambda - 0.7)(\lambda - 1.1)$$

$$\begin{bmatrix} 0.4 - 0.7 & 0.7 \\ -0.3 & 1.4 - 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -0.3 & 0.7 & | & 0 \\ -0.3 & 0.7 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{7}{3} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0.4 - 1.1 & 0.7 \\ -0.3 & 1.4 - 1.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -0.7 & 0.7 & | & 0 \\ -0.3 & 0.3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Appendix 2. Step 2b) Find P^{-1} .

$$\left[\begin{array}{cc|cc} 7 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{7} & \frac{1}{7} & 0 \\ 3 & 1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & \frac{4}{7} & -\frac{3}{7} & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 1 & -\frac{3}{4} & \frac{7}{4} \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} & \frac{7}{4} \end{array} \right]$$

$$P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 0.25 & -0.25 \\ -0.75 & 1.75 \end{bmatrix}$$

Appendix 3. Step 3b), part 2. Solve the differential equations you just found.

This one also needs the first-order linear DE process.

$$\begin{aligned} \frac{dy_2}{dt} - 1.1y_2 &= 1.75e^t \Rightarrow I(t) = e^{\int -1.1 dt} = e^{-1.1t} \Rightarrow e^{-1.1t} y_2 = \int 1.75e^t * e^{-1.1t} dt \\ \Rightarrow e^{-1.1t} y_2 &= \int \frac{7}{4} e^{-0.1t} dt = \frac{7}{4} * -\frac{10}{1} e^{-0.1t} + C_2 \Rightarrow y_2 = -\frac{35}{2} e^t + C_2 e^{1.1t} \end{aligned}$$

Appendix 4. If you have time you can check your equations by finding derivatives and plugging back into the original system of DEs.

$$x_1 = -\frac{70}{3} e^t + 7C_1 e^{0.7t} + C_2 e^{1.1t}$$

$$\frac{dx_1}{dt} = -\frac{70}{3} e^t + 4.9C_1 e^{0.7t} + 1.1C_2 e^{1.1t}$$

$$\begin{aligned} 0.4x_1 + 0.7x_2 &= -\frac{28}{3} e^t + 2.8C_1 e^{0.7t} + 0.4C_2 e^{1.1t} - 14e^t + 2.1C_1 e^{0.7t} + 0.7C_2 e^{1.1t} \\ &= \left(-\frac{28}{3} e^t - \frac{42}{3} e^t\right) + \left(2.8C_1 e^{0.7t} + 2.1C_1 e^{0.7t}\right) + \left(0.4C_2 e^{1.1t} + 0.7C_2 e^{1.1t}\right) \end{aligned}$$

$$x_2 = -20e^t + 3C_1 e^{0.7t} + C_2 e^{1.1t}$$

$$\frac{dx_2}{dt} = -20e^t + 2.1C_1 e^{0.7t} + 1.1C_2 e^{1.1t}$$

$$\begin{aligned} -0.3x_1 + 1.4x_2 + e^t &= 7e^t - 2.1C_1 e^{0.7t} - 0.3C_2 e^{1.1t} - 28e^t + 4.2C_1 e^{0.7t} + 1.4C_2 e^{1.1t} + e^t \\ &= \left(7e^t - 28e^t + e^t\right) + \left(-2.1C_1 e^{0.7t} + 4.2C_1 e^{0.7t}\right) + \left(-0.3C_2 e^{1.1t} + 1.4C_2 e^{1.1t}\right) \end{aligned}$$

Appendix 5. Step 5) (if needed) Substitute initial condition values to find C_1 and C_2 .

Suppose I have an initial population of 20 foxes and 50 rabbits. That is, at time $t = 0$, $x_1 = 20$ and $x_2 = 50$.

$$\begin{cases} x_1 = -\frac{70}{3} e^t + 7C_1 e^{0.7t} + C_2 e^{1.1t} \\ x_2 = -20e^t + 3C_1 e^{0.7t} + C_2 e^{1.1t} \end{cases} \Rightarrow \begin{cases} 20 = -\frac{70}{3} e^0 + 7C_1 e^0 + C_2 e^0 \\ 50 = -20e^0 + 3C_1 e^0 + C_2 e^0 \end{cases} \Rightarrow \begin{cases} \frac{130}{3} = 7C_1 + C_2 \\ 70 = 3C_1 + C_2 \end{cases}$$

$$\left[\begin{array}{cc|c} 7 & 1 & \frac{130}{3} \\ 3 & 1 & 70 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{7} & \frac{130}{21} \\ 3 & 1 & 70 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{7} & \frac{130}{21} \\ 0 & \frac{4}{7} & \frac{360}{7} \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{7} & \frac{130}{21} \\ 0 & 1 & 90 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{20}{3} \\ 0 & 1 & 90 \end{array} \right] \rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{20}{3} \\ 90 \end{bmatrix}$$

The particular solution is $x_1 = -\frac{70}{3} e^t + 7\left(-\frac{200}{3}\right) e^{0.7t} + 90e^{1.1t}$ and $x_2 = -20e^t + 3\left(-\frac{20}{3}\right) e^{0.7t} + 90e^{1.1t}$.