Nonlinear Differential Equations

1. **The Situation:** We've looked at linear systems - these are systems which have the form

$$\frac{dx_1}{dt} = Ax_1 + Bx_2 + C$$
$$\frac{dx_2}{dt} = Dx_1 + Ex_2 + F$$

with A, B, D, E constants and perhaps C, F function of t. We can solve these as we did in the previous section.

A nonlinear system is a system which is not of this form. An couple of examples would be

Example 1:

$$\frac{dx_1}{dt} = 0.4x_1 - 0.002x_1x_2$$
$$\frac{dx_2}{dt} = 0.3x_2 - 0.001x_1x_2$$

Example 2:

$$\frac{dx_1}{dt} = x_2^2 - x_1x_2 - x_2$$
$$\frac{dx_2}{dt} = 2x_1^2 + x_1x_2 - 7x_1$$

It is very difficult to solve nonlinear systems of differential equations and so we won't (whew!), but we will analyze them a little because they come up a lot in biology. Specifically we will look at two things:

- Equilibrium (stability) points and phase plane diagrams.
- Relating x_1 and x_2.

2. **Equilibrium (Stability) Points and Phase Plane Diagrams - Example 1**

Suppose x_1 represents the population of foxes and x_2 represents the population of rabbits, both after t weeks.

Question: Are any populations which are stable, meaning they don't change over time, meaning the derivatives are equal to 0?

Solution: If we set the derivatives in the system equal to 0:

$$0.4x_1 - 0.002x_1x_2 = 0$$
$$0.3x_2 - 0.001x_1x_2 = 0$$

The first is $0.002x_1(200 - x_2) = 0$ so $x_1 = 0$ or $x_2 = 200$. If $x_1 = 0$ then the second gives $0.3x_2 = 0$ and so $x_2 = 0$ and so we have $(0, 0)$. If $x_2 = 200$ then the second gives $60 - 0.2x_1 = 0$ and so $x_1 = 300$ and so we have $(300, 200)$.

So the first correponds to a stable population of 0 foxes and 0 rabbits - boring! The second corresponds to a stable population of 300 foxes and 200 rabbits - interesting! The nonzero pair is the *nontrivial equilibrium point*. From here on out we'll assume both x_1 and x_2 are positive!

Question: What happens *near* the equilibrium point?

Solution: Well what if $x_1 = 310$ and $x_2 = 190$? Are the populations increasing or decreasing? We could simply plug these values of x_1 and x_2 into the system:

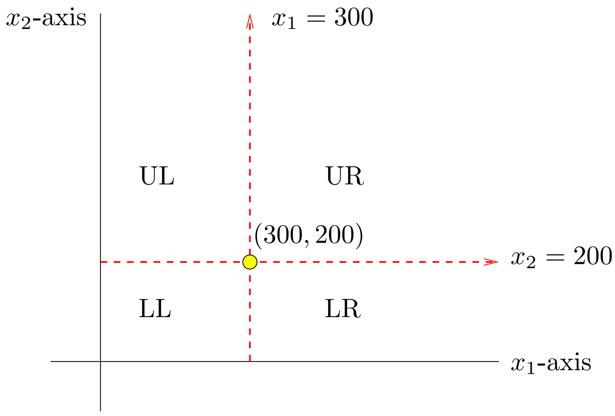
$$\frac{dx_1}{dt} = 0.4(310) - 0.002(310)(190) = 6.2$$
$$\frac{dx_2}{dt} = 0.3(190) - 0.001(310)(190) = -1.9$$

We see that the population of foxes is growing at 6.2 per week and the population of rabbits is shrinking at 1.9 per week.

Instead of repeating this for lots of different combinations of x_1 and x_2 we can draw a picture, called a *phase plane diagram*, to help us see. We know from precalculus that an expression will change from positive to negative or negative to positive only when it equals zero.

So first look at where $\frac{dx_1}{dt} = 0$ alone. This would be $0.4x_1 - 0.002x_1x_2 = 0$. Since $x_1 > 0$ we divide it out and so $x_2 = 200$. Then look at where $\frac{dx_2}{dt} = 0$ alone. This would be $0.3x_2 - 0.001x_1x_2 = 0$. Since $x_2 > 0$ we divide it out and so $x_1 = 300$. So in the first quadrant of the x_1x_2-plane draw the two lines $x_1 = 300$ and $x_2 = 200$. For good measure notice that the point where they meet is the nontrivial equilibrium point we discovered earlier.

These lines are the places where the derivatives can change from being positive to negative or vice versa.



Then we test a point in each of the four regions around the equilibrium point. In the picture we labelled these UR, UL, LL, LR (upper right, etc.)

We did $(310, 190)$ which is in the LR region and found that x_1 increases while x_2 decreases. This means we go right (x_1 increases) and down (x_2 decreases). To illustrate this we draw a small arrow going down and right in the LR region.

We do this for the other three regions:

We do $(310, 210)$ (UR)

$$\frac{dx_1}{dt} = 0.4(310) - 0.002(310)(210) = -6.2 \quad \text{Left and}$$
$$\frac{dx_2}{dt} = 0.3(210) - 0.001(310)(210) = -2.1 \quad \text{down.}$$

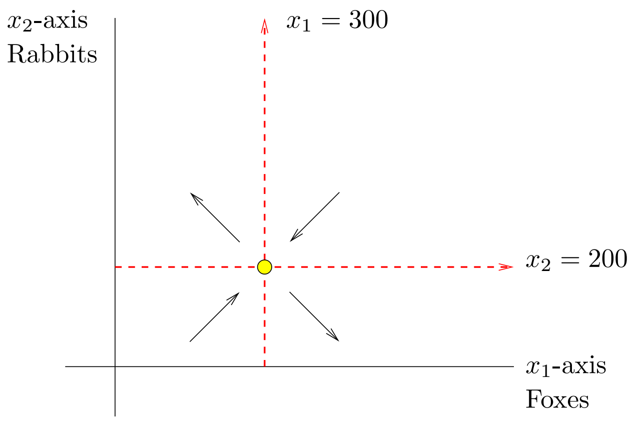
We do $(290, 190)$ (LL)

$$\frac{dx_1}{dt} = 0.4(290) - 0.002(290)(190) = 5.8 \quad \text{Right and}$$
$$\frac{dx_2}{dt} = 0.3(190) - 0.001(290)(190) = 1.9 \quad \text{up.}$$

We do $(290, 210)$ (UL)

$$\frac{dx_1}{dt} = 0.4(290) - 0.002(290)(210) = -5.8 \quad \text{Left and}$$
$$\frac{dx_2}{dt} = 0.3(210) - 0.001(290)(210) = 2.1 \quad \text{up.}$$

When all is said and done we have the following:



The arrows indicate what populations do in each region. The arrow in the UR region indicates that if both the populations of foxes is greater than 300 and the population of rabbits is greater than 200 they will both tend to decrease. On the other hand the the arrow in the UL region indicates that if the population of foxes is less than 300 and the population of rabbits is greater than 200 then the number of foxes will decrease while the number of rabbits will increase. This makes good sense if you consider it - fewer foxes mean less hunting of rabbits which means more rabbits.

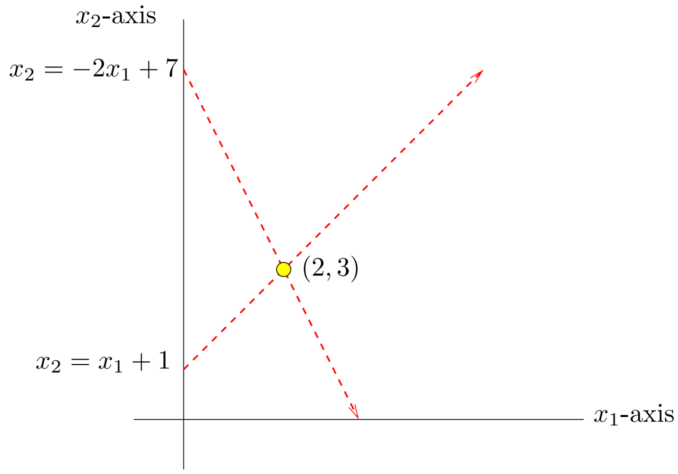
3. **Equilibrium (Stability) Points and Phase Plane Diagrams - Example 2**

Here is another example with a different picture. We'll be brief so you see this can be done efficiently.

$$\frac{dx_1}{dt} = x_2^2 - x_1x_2 - x_2$$
$$\frac{dx_2}{dt} = 2x_1^2 + x_1x_2 - 7x_1$$

First we find the equilibrium solution by setting both to zero. The first gives $x_2(x_2-x_1-1) = 0$ and ignoring $x_2 = 0$ gives $x_2 - x_1 - 1 = 0$. The second gives $x_1(2x_1 + x_2 - 7) = 0$ and ignoring $x_1 = 0$ gives $2x_1 + x_2 - 7 = 0$. Solving these by substitution (or whatever) gives $x_1 = 2$ and $x_2 = 3$. The equilibrium point is then $(2, 3)$.

To draw the phase plane we notice that when we set $\frac{dx_1}{dt} = 0$ we got the line $x_2 = x_1 + 1$ (this is a line in the x_1x_2-plane) and when we set $\frac{dx_2}{dt} = 0$ we got the line $x_2 = -2x_1 + 7$ (another line). Let's plot these along with the equlibrium point.



Then we pick a sample point in each region:

Check $(2, 4)$, above the equilibrium point

$$\frac{dx_1}{dt} = (4)^2 - (2)(4) - (4) = + \quad \text{Right and}$$
$$\frac{dx_2}{dt} = 2(2)^2 + (2)(4) - 7(2) = + \quad \text{up.}$$

Check $(3,3)$, to the right of the equilibrium point

$$\frac{dx_1}{dt} = (3)^2 - (3)(3) - (3) = - \quad \text{Left and}$$
$$\frac{dx_2}{dt} = 2(3)^2 + (3)(3) - 7(3) = + \quad \text{up.}$$

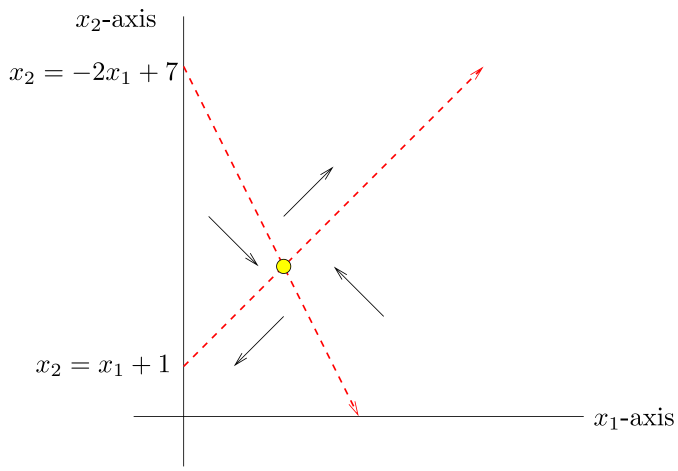
Check $(2,2)$, below the equilibrium point

$$\frac{dx_1}{dt} = (2)^2 - (2)(2) - (2) = - \quad \text{Left and}$$
$$\frac{dx_2}{dt} = 2(2)^2 + (2)(2) - 7(2) = - \quad \text{down.}$$

Check $(1,3)$, to the left of the equilibrium point

$$\frac{dx_1}{dt} = (3)^2 - (1)(3) - (3) = + \quad \text{Right and}$$
$$\frac{dx_2}{dt} = 2(1)^2 + (1)(3) - 7(1) = - \quad \text{down.}$$

So when all is said and done we have



So if this were a fox and rabbit problem we'd see that if there were $x_1 = 5$ foxes and $x_2 = 3$ rabbits (to the right of the equlibrium point) then the number of foxes would decrease while the number of rabbits increased.

4. **Relating x_1 and x_2:** Even though we can't solve the system we can do something. Consider the first example. It might help you here to use x and y in place of x_1 and x_2 so we'll do that. First we factor a bit:

$$\frac{dx}{dt} = 0.4x - 0.002xy = x(0.4 - 0.002y)$$
$$\frac{dy}{dt} = 0.3y - 0.001xy = y(0.3 - 0.001x)$$

and then we pull a fancy trick. Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (this is actually from the chain rule $\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt}$ we have:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(0.3 - 0.001x)}{x(0.4 - 0.002y)}$$

which is a separable differential equation! Wow! Let's solve it. Keep in mind that both x and y are positive.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y(0.3 - 0.001x)}{x(0.4 - 0.002y)} \\
\frac{0.4 - 0.002y}{y}\ dy &= \frac{0.3 - 0.001x}{x}\ dx \\
\int \frac{0.4 - 0.002y}{y}\ dy &= \int \frac{0.3 - 0.001x}{x}\ dx \\
\int \frac{0.4}{y} - 0.002\ dy &= \int \frac{0.3}{x} - 0.001\ dx \\
0.4\ln y - 0.002y &= 0.3\ln x - 0.001x + C
\end{aligned}$$

If we go back to x_1 and x_2 we have

$$0.4\ln x_2 - 0.002x_2 = 0.3\ln x_1 - 0.001x_1 + C$$

Now then, solving for either variable is impossible but let's appreciate what we've got. This equation is an equation which relates the number of foxes to the number of rabbits even though we can't find either as a function of time.

If we've given a pair we can still find C as before. For example if $x_1 = 100$ and $x_2 = 200$ then

$$0.4\ln 200 - 0.002(200) = 0.3\ln 100 - 0.001(100) + C$$

and so

$$C = 0.4\ln 200 - 0.3\ln 100 - 0.3$$

and so

$$0.4\ln x_2 - 0.002x_2 = 0.3\ln x_1 - 0.001x_1 + 0.4\ln 200 - 0.3\ln 100 - 0.3$$

5. **Summary:**

   (a) To find the equilibrium point and phase-plane diagram:

      i. Set the derivatives equal to 0. Assume x_1 and x_2 are nonzero. Each will give you a line.

      ii. Solve them as a system, this will give you a point - the equilibrium point.

      iii. Plot the lines and the point.

      iv. Pick a point in each of the four regions and plug it into the system. This will give you an arrow for each region.

      v. Understand what you've got, especially in real-world problems!

   (b) To find the relationship between x_1 and x_2:

      i. Set $\frac{dx_2}{dx_1}$ equal to $\frac{dx_2/dt}{dx_1/dt}$. For our examples this will be separable.

      ii. Separate and solve.

      iii. Understand what you've got, especially in real-world problems!