Calculus 131, section 12.2 Introduction to Probability

notes by Tim Pilachowski

We now move into probability, the realm of "What could happen?" and "How likely is it to happen?"
The set of "What could happen?" is called the **sample space**. The elements in the sample space are the possible **outcomes**.

An **event** is a subset of the sample space, and will incorporate one or more of the outcomes. Two events F and G are **mutually exclusive** if their intersection is the null set.

The **probability** of an event $E = \dfrac{\text{number of ways } E \text{ can happen}}{\text{number of possible outcomes}}$.

If the outcomes in sample space S are equally likely, then $P(E) = \dfrac{n(E)}{n(S)}$.

Example A: You toss two coins.

sample space:

(For events and probabilities below, use S_2.)

E_1 = both coins are heads =

E_2 = one coin is heads and the other is tails =

probability that both coins are heads = $P(E_1)$ =

P(one coin is heads and the other is tails) = $P(E_2)$ =

Example B: You toss two dice.

sample space: outcomes are written as (value of first die, value of second die)

S = { (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1),
(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2),
(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3),
(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4),
(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5),
(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6) }

E_1 = at least one of the dice is a 1 =

E_2 = the sum of the two dice is 9 =

E_3 = at least one of the dice is a 1 ***and*** the sum of the two dice is 9 =

Events E_1 and E_2 are **mutually exclusive.**

P(at least one of the dice is a 1) = $P(E_1)$ =

P(the sum of the two dice is 9) = $P(E_2)$ =

P(at least one of the dice is a 1 ***and*** the sum of the two dice is 9) = $P(E_3)$ =

Example C: You pick a card from a standard deck of 52 cards. [4 suits: spades (S), hearts (H), diamonds (D), clubs (C); 13 cards in each suit: ace (A), king (K), queen (Q), jack (J), 10, 9, 8, 7, 6, 5, 4, 3, 2]

sample space: outcomes are written as value-suit
$n(S) = 52$. S = { A-S, A-H, A-D, A-C, K-S, K-H, K-D, K-C, Q-S, Q-H, Q-D, Q-C, … , 2-S, 2-H, 2-D, 2-C }.
The 52 outcomes are equally likely.

E_1 = you pick an Ace =

E_2 = you pick a spade =

E_3 = you pick the Ace of spades = the card is both an Ace ***and*** a spade =

E_4 = you pick an Ace ***or*** a spade =

P(you pick an Ace) = $P(E_1)$ =

P(you pick a spade) = $P(E_2)$ =

P(you pick the Ace of spades) = P(the card is an Ace ***and*** it is a spade) = $P(E_3)$ =

P(you pick an Ace ***or*** a spade) = $P(E_4)$ =

This formula is important: the **union rule for probability,** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorems about probability:

E_5 = you pick a card that is neither an Ace nor a spade

Event E_5 is the **complement** of one used earlier, E_4 = "you pick an Ace ***or*** a spade"

Rather than calculating $P(E_5)$ directly, we'll calculate a complement.

$P(E_5)$ = P(you pick a card that is neither an Ace nor a spade) =

Example D: Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick three blocks without replacement.

One way to create this sample space is to designate the block by color and number: B1, B2, B3, Y1, Y2. Then we'd have S = { B1-B2-B3, B1-B2-Y1, B1-B2-Y2, B1-Y1-B2, B1-Y1-B3, B1-Y1-Y2, etc., etc., etc., … }

While the outcomes would be equally likely, the process would be extremely cumbersome. Imagine having 333 blue and 222 yellow!

Instead, we'll consider just blue and yellow colors, and construct a tree diagram. The tree diagram for randomly picking three blocks, one at a time, would look like this:

Note that for the lowest branch, once both yellow blocks have been picked, the only possibility left is a blue block. Also note, these are not equally likely outcomes. (In Lecture 12.3, we'll consider the probabilities involved in this trial.)

E_1 = all three blocks are blue =

E_2 = two are blue and one is yellow =

E_3 = one is blue and two are yellow =

E_4 = all three blocks are yellow =

So far, the probabilities encountered have been theoretical. An empirical probability is based on observed data. The **frequency** of an event or outcome E is how many times it was observed to occur. The **relative frequency** is $\frac{\text{frequency of } E}{\text{number of observations}} = \frac{n(E)}{n(S)}$. In this type of situation, the probability of an event E is

$P(E)$ = value to which the relative frequency stabilizes with an increasing number of trials.

Example E: A hospital records the number of days each ICU patient stays in intensive care. $S = \{ 1, 2, 3, \ldots \}$. Out of 1247 ICU patients in the last 15 years, 536 stayed in ICU two weeks or less. If a patient is selected at random, what is the probability that she or he will stay in ICU two weeks or less?

Example F: In 2009, households were surveyed about health insurance coverage, with the following results.

	Age 18-64	Age < 18	Totals
Public Plan	10286	10771	
Private Plan	47000	15914	
Uninsured	14144	1885	
Totals			100000

DATA SOURCE: CDC/NCHS, National Health Interview Survey, 1997-2009, Family Core component. Data are based on household interviews of a sample of the civilian noninstitutionalized population. (caveat: The report from which Example G was developed did not provide the numbers in the table above. Rather they were calculated from the given data, using an entirely fictional sample size of 100,000.)

E_1 = individual is under 18; $P(E_1)$ =

E_2 = individual has health insurance; $P(E_2)$ =

E_3 = individual is under 18 ***and*** has health insurance; $P(E_3)$ =

E_4 = individual is under 18 ***or*** has health insurance; $P(E_4)$ =

Information not included in the syllabus for Math 131, but which might be of interest to you.
Although "odds" and "probability" are often used interchangeably in everyday conversation, they do not have the same mathematical meaning. Rather, odds are calculated as a ratio of probabilities. For example, if there is a 35% chance of rain, then the odds would be calculated as follows:

$$\text{odds in favor of rain} = \frac{\text{probability it will rain}}{\text{probability it won't rain}} = \frac{0.35}{0.65} = \frac{7}{13}, \text{ or odds of 7 to 13}$$

$$\text{odds against rain} = \frac{\text{probability it won't rain}}{\text{probability it will rain}} = \frac{0.65}{0.35} = \frac{13}{7}, \text{ or odds of 13 to 7}$$

Odds are most often used in gambling situations. See pages 681-682 of your text for more information.