Calculus 131, section 12.4 Discrete Random Variables

notes by Tim Pilachowski

Do you remember how to calculate an average? The word "average", however, has connotations outside of a strict mathematical definition, so mathematicians have a different name: the ***expected value***. (Statisticians will tend to use "expected value" and "mean" interchangeably.)

Example A: Suppose we measure the heights of 25 people to the nearest inch and get the following results:

height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2

What is the expected value for height? *answer*: 65.88 in.

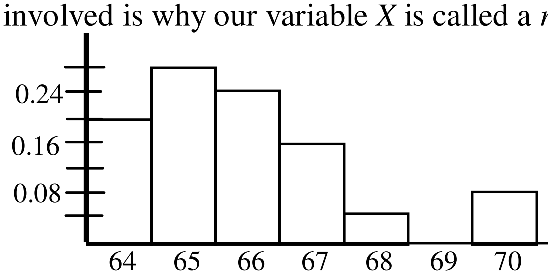
In statistics, the formula "sum of [value times probability]" is usually called the "mean".

$\mathrm{E}(X) = \mu = x_1\, P(x_1) + x_2\, P(x_2) + x_3\, P(x_3) + \ldots + x_n\, P(x_n) = \sum_x x\, P(x)$ where x_k is a possible value for a random variable X and $P(x_k)$ is its probability

We can think of probability as an area. In the chart below,

{area of each bar} = {percentage of people having that height}
= {relative frequency of that height}
= {probability of a person picked at random having that height}.

The element of chance/randomness involved is why our variable X is called a *random variable*.



Note that the areas of the rectangles must add up to 1, because they represent the probabilities for *every* outcome in a sample space.

Some vocabulary: A bar graph (like the one above) which shows the probabilities for a discrete random variable X is called a **histogram**. The function $P(x) = P(X = x)$ is called (what a surprise!) a **probability function**.

While we're here, we'll introduce ourselves to formulae for *variance* and *standard deviation*.

$$\operatorname{Var}(X) = (x_1-\mu)^2 P(x_1) + (x_2-\mu)^2 P(x_2) + (x_3-\mu)^2 P(x_3) + \ldots + (x_n-\mu)^2 P(x_n) = \sum_x (x_k-\mu)^2 P(x_k)$$

standard deviation of $X = \sigma = \sqrt{\operatorname{Var}(X)}$

Variance and standard deviation are both measures of how much the amounts (x_k) vary (or deviate) from the mean ($\mathrm{E}(X) = \mu$).

Example A revisited: We measure the heights of 25 people to the nearest inch and get the following results:

x = height (in.)	64	65	66	67	68	69	70
$P(x)$ = Probability							

More vocabulary: A table (like the one above) which shows the probabilities for a discrete random variable X is called a **probability distribution**. Note that the sum of the probabilities must equal 1, because they represent the probabilities for *every* outcome in the sample space.

What is the expected value, E(X), for height? What are Var(X) and the standard deviation for these heights?
answers: 65.88 in., ≈ 2.6656, ≈ 1.6327

Example B: If one rolls two dice, what are E(X), Var(X) and the standard deviation for the sum?

x = sum	2	3	4	5	6	7	8	9	10	11	12	
$P(x)$												Total = **1**

answers: 7, ≈ 5.8333, ≈ 2.4152

Example B extended: You have the chance to play a game in which you roll two dice. You win \$1 if the sum is even, and lose \$2 if the sum is odd. What are E(X), Var(X) and the standard deviation for this game? What is your expected win or loss if you roll the dice 100 times?

sum	2	3	4	5	6	7	8	9	10	11	12	
x = winnings	1	–2	1	–2	1	–2	1	–2	1	–2	2	
$P(x)$												Total **= 1**

answers: – 0.5, 2.25, 1.5, – \$50

Example C: Five coins are tossed. If 0, 1, or 2 heads come up, the player wins nothing. If 3 heads come up the player wins \$2. If 4 heads come up the player wins \$5, and if 5 heads come up the player wins \$10. What fee charged to the player would make this a fair game? If the house charges \$2 to play and 1000 people play, about how much profit should the house expect to make?

no. of heads	0	1	2	3	4	5	
x = winnings							
$P(x)$							Total **= 1**

answers: ≈ \$1.72, \$281.50

Example D: Insurance companies use actuarial data to set rates for policies. Collected data indicate that, on a \$1000 policy, an average of 1 in every 100 policy holders will file a \$20,000 claim. An average of 1 in every 200 policy holders will file a \$50,000 claim. An average of 1 in every 500 policy holders will file a \$100,000 claim. What are E(X), Var(X) and the standard deviation for the value of a policy to the company? If the company sells 100,000 policies, what is its expected profit or loss?

x = value of a policy					
$P(x)$					Total **= 1**

answers: \$350, 36077500, ≈ 6006.4549, \$35,000,000

Example E: In 1953, French economist Maurice Allais studied how people assess risk by giving them two decisions to make.

1) Choose between A = { 100% chance of getting \$1 million } and B = { 10% chance of getting \$2.5 million, 89% chance of getting \$1 million, 1% chance of getting nothing }.

2) Choose between A = { 11% chance of getting \$1 million, 89% chance of getting nothing } and B = {10% chance of getting \$2.5 million, 90% chance of getting nothing}.

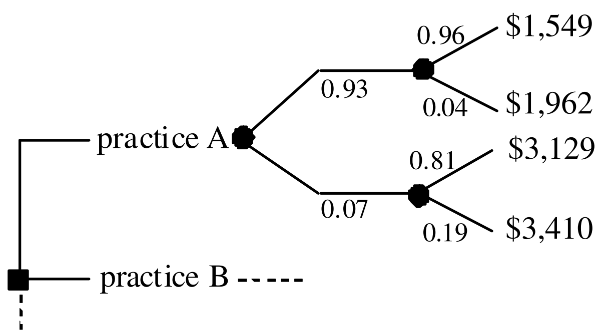
Allais found that most people chose A for decision 1) and B for decision 2). Use a decision tree and expected value to determine whether these choices are supported by the numbers.

answer: \$1 million, \$1.14 million, \$110,000, \$250,000

In real-world business, decisions will be needed for ongoing production and marketing strategies. In medicine, treatment decisions need to be made, using evaluations of different options and probabilities of success or failure.

In text exercises, you are given the decision tree and associated probabilities. In Example D, we had no conditional probabilities, i.e. we only had to consider one probability at a time. In a case where there are conditional probabilities, use the multiplication rule.

Example text #20:



For the first option, the probability $P(x) = 0.93\ (0.96) = 0.8928$. This would then be multiplied times $x = \$1549$ in the calculation of expected cost.