Calculus 131, section 13.3 Special Probability Density Functions

notes by Tim Pilachowski

13.2 & 13.3 Example A: Find the expected value and standard deviation for the uniform probability density function $f(x) = \frac{1}{10},\ 0 \le x \le 10$.

This function is called *uniform* because it has a constant value across its domain.
We found the expected value and variance in Lecture 13.2, Example A.

$$\mathrm{E}(X) = \int_0^{10} x * \frac{1}{10}\, dx = 5 \qquad\qquad \mathrm{Var}(X) = \int_0^{10} x^2 * \frac{1}{10}\, dx - [5]^2 = \frac{25}{3}$$

standard deviation =

Your text gives formulae for mean and standard deviation. Rather than memorize yet another set of formulae, my recommendation is that you do the integral work.

13.2 & 13.3 Example D. For a particular machine, its useful lifetime is modeled by $f(t) = 0.1e^{-0.1t}$, $0 \le t \le \infty$. Find μ and σ.

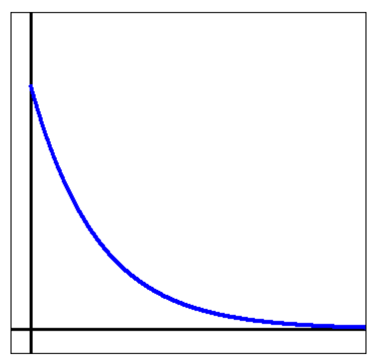
This should look familiar. As Example D in Lecture 13.1, we verified that *f* is a probability density function, then found various probabilities.

In Lecture 13.2, Example D, we had to go through a very extensive integration by parts to find expected value, variance and standard deviation.

$$\mu = \mathrm{E}(X) = \lim_{b \to \infty} \int_a^b t\ 0.1e^{-0.1t}\, dt = 10 \qquad \mathrm{Var}(X) = \lim_{b \to \infty} \int_0^b t^2\left(0.1e^{-0.1t}\right) dt - \left[10^2\right] = 100$$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{100} = 10$$

Exponential probability density functions have the general form $f(t) = ae^{-at}$, and most often describe the distance between events with uniform distribution in time.



The exponential distribution is used widely in analysis of reliability, which deals with the amount of time a product lasts (as in Example D). The length of time a long distance phone call lasts follows an exponential distribution—there are more phone calls that last a shorter amount of time and fewer calls that last a long time. Another example is the amount of money a customer spends for one trip to the supermarket—there are more people that spend less money and fewer people that spend large amounts of money.

Exponential probability distributions have very nice characteristics that make them easy to work with. If we duplicated the work above and replaced the coefficient 0.1 with *a*, and replaced its reciprocal 10 with $\frac{1}{a}$ (as your text does), we would get the following results: For $f(x) = ae^{-ax}$, $\mu = \frac{1}{a}$, $\mathrm{Var}(X) = \frac{1}{a^2}$ and $\sigma = \frac{1}{a}$.

I recommend memorizing these.

Example D revisited. For a particular machine, its useful lifetime (random variable *X*) is modeled by $f(t) = 0.1e^{-0.1t}$, $0 \le t \le \infty$. Find the expected lifetime of the machine, along with variance and standard deviation.

Example B. Given the exponential probability density function $f(x) = 3e^{3x}$, $0 \le x \le \infty$, find μ and σ.

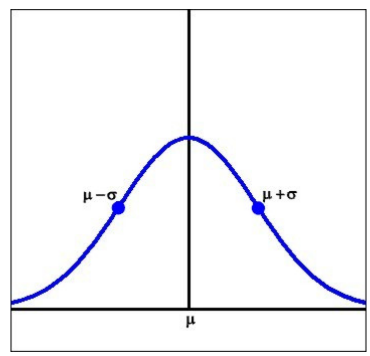
Example C: The mean time (expected value) between hits at a website is 0.5 seconds. a) Find the exponential probability density function $f(x)$ for random variable X = time between hits. b) Find Var(X). c) Find standard deviation. d) Find the probability that the time to the next hit is between 0.25 and 0.75 seconds. e) Find the probability that the time to the next hit is less than 0.3 seconds. f) Find the probability that the time to the next hit is greater than 1 second.

answers d) – f): $-e^{-1.5} + e^{-0.5} \approx 0.3834$; $-e^{-0.6} + 1 \approx 0.4512$; $e^{-2} \approx 0.1353$

Example E: Scientists have estimated that a "doomsday meteor" will hit the earth every 100 million years on average. (Source: NASA International Near-Earth-Object Detection Workshop (*The New York Times,* January 25, 1994, p. C1.) a) Assuming an exponential probability distribution, what is the approximate probability that the earth will be hit by a doomsday meteor at least once during the next millenium? b) What is the approximate probability that the earth has been hit by a doomsday meteor at least once since the appearance of life (about 4 billion years ago)? *answers*: ≈ 0.000 01; ≈ 1

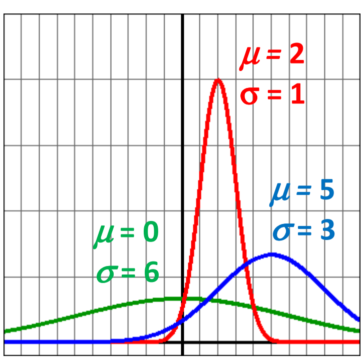
Another often-useful probability density function is the normal density function, which graphs as the familiar bell-shaped curve. The generic format is

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where $E(X) = \mu$, $\text{Var}(X) = \sigma^2$, and standard deviation $=\sqrt{\text{Var}(X)}=\sigma$.



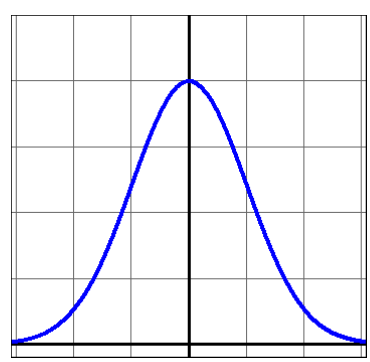
The graph of a normal curve is symmetric with respect to the line $x=\mu$, and has points of inflection at $x=\mu-\sigma$ and $x=\mu+\sigma$.

Side note: More than 68% of a *normally distributed* population will fall within $\pm 1\sigma$ of the mean, more than 95% will fall within $\pm 2\sigma$, and more than 99.5% within $\pm 3\sigma$.



Differences in the means result in shifts left and right. A smaller standard deviation will result in a taller, more narrow "bell". Each curve is symmetric about the mean. Note that in all three cases, probabilities beyond $\mu \pm 3\sigma$ become so small as to usually be considered insignificant.

In the special case where $E(X) = \mu = 0$ and standard deviation $= \sigma = 1$, we get what is called the standard normal distribution, $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, with its graph called the standard normal curve.

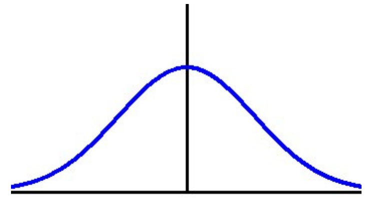


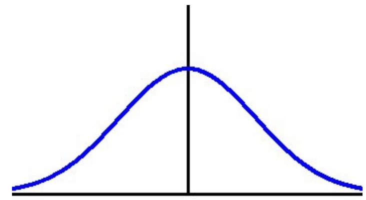
There is no method for integrating to find area under the curve (and thus probabilities) of this standard normal probability density function, but a calculation using a Taylor polynomial (covered in Math 141) has been done to construct tables of values such as the one found in the Appendix of your text.

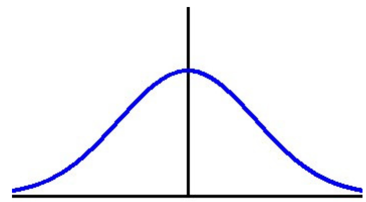
For a standardized random variable Z, this text's normal distribution table gives us

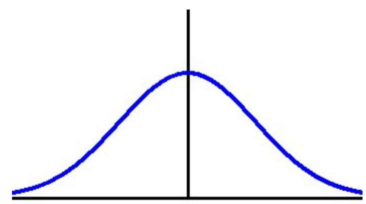
Area under the curve from "forever left" to $z = P(-\infty < z \leq \text{specified value})$.

Examples F: Given a standard normal random variable z, find a) $P(Z \leq 1.23)$, b) $P(0 \leq Z \leq 1.23)$, c) $P(-1.23 \leq Z \leq 0)$, d) $P(-2.14 \leq Z \leq 1.23)$. *answers*: 0.8907; 0.3907; 0.3907; 0.8745

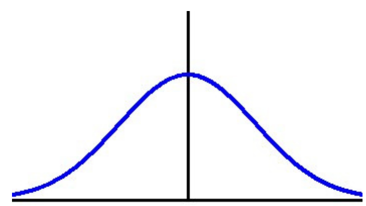


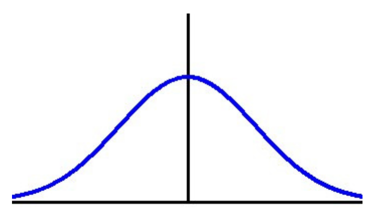


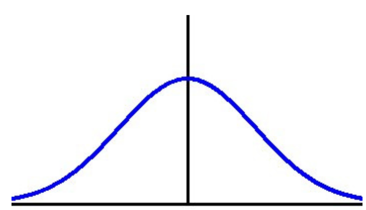




Examples G: Given a standard normal random variable Z, find a) the value z that marks the lowest 10%, b) the value z that marks the top 5%, c) the values z that mark the middle 50%.
answers: –1.28; 1.645; do as text practice exercise

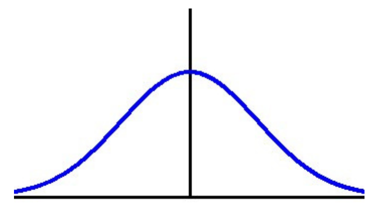


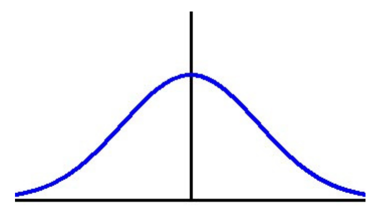




Rather than approximate values for every possible normal density function, the common practice is to convert everything to a standard normal distribution and use the same normal distribution table over and over. In the science of statistics, where things such as sampling distributions are normally distributed, data will be converted to the standard normal random variable Z, where $Z = \dfrac{X - \mu}{\sigma}$.

Examples H: (source http://www.cdc.gov/nchs/data/ad/ad347.pdf) "During the last 2 decades the prevalence of obesity and overweight has increased in the United States. This is in part due to a shift in the distribution of BMI (weight in kilograms divided by height in meters squared) of the entire population…Between the early1960s and 1999–2002 mean BMI for men 20–74 years of age increased from just over 25 to almost 28. Similarly, for women mean BMI increased from almost 25 to just over 28." Distribution is approximately normal. Using data from Table 10 of this report, in 1999-2002 mean BMI for adults 20 years and over = 27.95, with a standard deviation of 0.15. a) Find the probability that an adult selected at random has a BMI between 27.7 and 28.3. b) Find the probability that an adult selected at random has a BMI greater than 28.3. c) Find the BMI value that marks the top 5%.
answers: 0.9426; 0.0099; 28.20





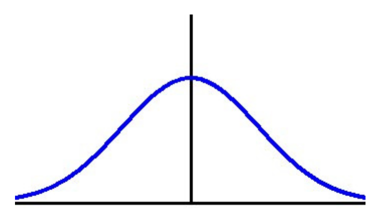


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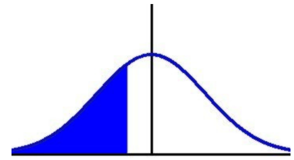
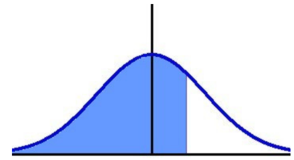
Table 2 Area Under a Standard Normal Curve to the Left of z ($z < 0$)



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Appendix A, page A-3

Table 2 Area Under a Standard Normal Curve to the Left of z ($z > 0$)



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998