Math 131 Exam 2 Sample 1 <u>SOLUTIONS</u> **Fall 2010**

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1. (a) Given $f(x, y, z) = x^2 y - x^3 z + x \ln y$ find $f(-2, 1, 3)$.

$$f(-2,1,3) = (-2)^2(1) - (-2)^3(3) + (-2)\ln(1) = (4)(1) - (-8)(3) + (-2)(0) = 4 + 24 + 0 = 28$$

1. (b) $\int_0^6 \int_0^{x^2} (x+y)\, dy\, dx$

$$\int_0^6 \left[xy + \frac{1}{2}y^2 \right]_0^{x^2} dx = \int_0^6 \left[x^3 + \frac{1}{2}x^4 \right] - [0]\, dx = \left[\frac{1}{4}x^4 + \frac{1}{10}x^5 \right]_0^6 = \left[\frac{1}{4}(6^4) + \frac{1}{10}(6^5) \right] = 1101.6$$

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2 (a) The surface area of a human body (in m^2) is approximately $A = 0.202 W^{0.425} H^{0.725}$, where *W* is the weight of the person in kilograms and *H* is the height in meters. Find an equation that represents how a person's surface area will change as her or his weight changes, assuming that height remains constant.

$$\frac{\partial A}{\partial W} = A_W = 0.202 * 0.425 W^{0.425-1} H^{0.725} = 0.08525 W^{-0.575} H^{0.725}$$

2. (b) For $g(x) = xy - e^{xy}$ find g_x and g_{xy}.

$$g_x = \frac{\partial}{\partial x}\left(xy - e^{xy}\right) = y - ye^{xy}$$

In the step below, note the use of the product rule for $\frac{\partial}{\partial y}\left(-ye^{xy}\right)$.

$$g_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial g}{\partial x}\right] = \frac{\partial}{\partial y}\left[y - ye^{xy}\right] = 1 - \left(xye^{xy} + e^{xy}\right) \text{ or } 1 - xye^{xy} - e^{xy}$$

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3. (a) Let $f(x, y) = x^4 - 10xy + \frac{1}{2}y^2 - 7$. It can be shown that the first partial derivatives of $f(x, y)$ are zero at (5, 50). Use the second derivative test to determine the nature of the function at this point. (Write the appropriate term: *relative maximum*, *relative minimum*, *neither a max nor a min (saddle point), or inconclusive*.)

Compute first partials. $f_x = 4x^3 - 10y$, $f_y = -10x + y$

Compute three second partials (based on the first partials) $f_{xx} = 12x^2$; $f_{yy} = 1$; $f_{xy} = -10$

Write $D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})^2 = 12x^2 - 100$.

Observe that $D(5,50) = 12(5^2) - 100$ is positive and indicates a max/min

Observe that $f_{xx}(5, 50) = 12(5^2) > 0$ and conclude that $f(x,y)$ has a relative minimum at (5,50).

3. (b) Solve the differential equation $(t^2+1)\frac{dy}{dt} = t(y+1)$. ***methods: separation of variables, substitution***

$$\frac{1}{y+1}\frac{dy}{dt} = \frac{t}{t^2+1} \Rightarrow \int \frac{1}{y+1}\, dy = \int \frac{t}{t^2+1}\, dt \Rightarrow \text{for the right side}: u = t^2+1, \quad du = 2t\, dt, \quad \frac{1}{2}du = 2t\, dt$$

$$\Rightarrow \ln|y+1| = \frac{1}{2}\int \frac{1}{u}\, du \Rightarrow \ln|y+1| = \frac{1}{2}\ln|u| + C$$

$$\Rightarrow y + 1 = e^{\frac{1}{2}\ln(t^2+1) + C} = e^C e^{\ln(t^2+1)^{1/2}} = C\sqrt{t^2+1} \Rightarrow y = C\sqrt{t^2+1} - 1$$

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4. (a) The temperature of a frozen pizza rises at a rate expressed by the differential equation $y' = 9(400 - y)$. A pizza with a temperature of 30° F is put into the oven. Solve to find an equation which expresses the pizza's temperature y as a function of time t in minutes.

$$y' = 9(400-y) \quad \Rightarrow \quad \int \frac{1}{400-y}\,dy = \int 9\,dt \quad \Rightarrow \quad -\ln(400-y) = 9t + C \quad \Rightarrow \quad \ln(400-y) = C - 9t$$

$$\Rightarrow \quad 400 - y = e^{C-9t} = Ce^{-9t} \quad \Rightarrow \quad -y = Ce^{-9t} - 400 \quad \Rightarrow \quad y = 400 - Ce^{-9t} \quad \Rightarrow \quad 30 = 400 - Ce^{0}$$

$$\Rightarrow \quad C = 370 \quad \Rightarrow \quad y = 400 - 370e^{-9t}$$

4. (b) A population of wolverines is modeled by the differential equation $\frac{dy}{dt} = 0.02y + 16t$. Solve to find an equation to represent population y at time t (in years) given an initial population of 10,000 wolverines.

method: first –order linear process

$$\frac{dy}{dt} - 0.02y = 16t \quad \Rightarrow \quad I(t) = e^{\int -0.02\,dt} = e^{-0.02t} \quad \Rightarrow \quad e^{-0.02t}\frac{dy}{dt} - 0.02e^{-0.02t}y = 16te^{-0.02t}$$

$$\Rightarrow \quad \int \frac{d}{dt}\left[e^{-0.02t}y\right]dt = \int 16te^{-0.02t}\,dt \quad \Rightarrow \quad dv = e^{-0.02t}\,dt,\ v = -50e^{-0.02t},\ u = 16t,\ du = 16\,dt$$

$$\Rightarrow \quad e^{-0.02t}y = \int 16te^{-0.02t}\,dt = 16t\left(-50e^{-0.02t}\right) - \int -50e^{-0.02t}(16)\,dt = -800te^{-0.02t} - 40000e^{-0.02t} + C$$

$$\Rightarrow \quad y = -800t - 40000 + Ce^{0.02t} \quad \Rightarrow \quad 10000 = -800(0) - 40000 + Ce^{0.02(0)} \quad \Rightarrow \quad C = 50000$$

$$\Rightarrow \quad y = -800t - 40000 + 50000e^{0.02t}$$

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5. (a) Let $f(t)$ be the solution to $y' = y + 4t^2,\quad y(1) = 6$. Use Euler's method with $n = 2$ to estimate $f(2)$.

$$h = \frac{2-1}{2} = \frac{1}{2},\quad t_0 = 1,\quad y_0 = 6,\quad y' = g(1,6) = 6 + 4\left(1^2\right) = 10$$

$$\Rightarrow t_1 = \frac{3}{2},\quad y_1 = 6 + 10\left(\frac{1}{2}\right) = 11,\quad y'\left(\frac{3}{2}, 11\right) = 11 + 4\left(\frac{3}{2}\right)^2 = 20$$

$$\Rightarrow t_2 = 2,\quad f(2) \cong y_2 = 11 + 20\left(\frac{1}{2}\right) = 21$$

(b) Given the system of linear differential equations, $\frac{dx_1}{dt} = 3x_1 - 2x_2 \qquad \frac{dx_2}{dt} = x_1 + e^t$, a student has found the necessary eigenvalues and eigenvectors to be $\lambda_1 = 1 \Rightarrow \begin{bmatrix}1\\1\end{bmatrix}$ and $\lambda_1 = 2 \Rightarrow \begin{bmatrix}2\\1\end{bmatrix}$. Do the next steps of the solution process to answer $\frac{dy_1}{dt} = ?$ and $\frac{dy_2}{dt} = ?$ Do not solve these two DEs!

the student's preliminary work to identify M and find eigenvalues and eigenvectors:

$$M = \begin{bmatrix}3 & -2\\1 & 0\end{bmatrix} \qquad M - \lambda I = \begin{bmatrix}3-\lambda & -2\\1 & -\lambda\end{bmatrix} \qquad \det(M - \lambda I) = (3-\lambda)(-\lambda) - (-2)(1) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

$$\begin{bmatrix}3-1 & -2\\1 & 0-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c}2 & -2 & 0\\1 & -1 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & -1 & 0\\1 & -1 & 0\end{array}\right] \quad \Rightarrow \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix} 3-2 & -2 \\ 1 & 0-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

the solution to the sample exam question: identify D and P, then find P^{-1}, then find $(y_1)'$ and $(y_2)'$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \qquad P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{dy_1}{dt} \\ \dfrac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} \Rightarrow \begin{matrix} \dfrac{dy_1}{dt} = y_1 + 2e^t \\ \dfrac{dy_2}{dt} = 2y_2 - e^t \end{matrix} \quad or \quad \begin{matrix} \dfrac{dy_1}{dt} - y_1 = 2e^t \\ \dfrac{dy_2}{dt} - 2y_2 = -e^t \end{matrix}$$

See Exam 2 Sample 2 answer key for the rest of the solution process.
