Math 131 Exam 2 Sample 2 SOLUTIONS **Fall 2010**

1. (a) The number of cows that can graze on a ranch is approximated by $f(x, y) = 9x + 5y - 5$, where x is the number of acres of grass and y the number of acres of alfalfa. First find $f(40, 75)$, then write a sentence explaining what your result means.

$f(40, 75) = 9(40) + 5(75) - 5 = 360 + 375 - 5 = 730$
When a ranch has 40 acres of grass and 75 acres of alfalfa, then 730 cows can graze.

1. (b) Find and categorize all relative maximum or minimum points of $g(x, y) = -x^2 - y^2 + 6x + 8y - 21$. You must show appropriate work to justify your conclusion.

$g_x = -2x + 6$, which equals 0 where $x = 3$. $g_y = -2y + 8$, which equals 0 where $x = 4$. So there is only one possible relative maximum or minimum. Apply the second derivative test to determine which it is.

$$g_{xx} = \frac{\partial}{\partial x}(-2x+6) = -2, \quad g_{yy} = \frac{\partial}{\partial y}(-2y+8) = -2, \quad g_{xy} = \frac{\partial}{\partial y}(-2x+6) = 0$$

So $D(x, y) = (-2)(-2) - 0^2 = 4 > 0$ for all values of x and y, including $x = 3$ and $y = 4$.
Since $f_{xx} = -2 < 0$ for all values of x and y, including $x = 3$ and $y = 4$, we conclude $f(3, 4)$ is a maximum.
Since there are no other possible extremes, this must be an absolute maximum.

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2. (a) Given $h(x, y, z) = x^2 y - x^3 z + x \ln y$ find $h_y(-2, 1, 3)$.

$$h_y = x^2 * \frac{\partial}{\partial y}(y) - x^3 z * \frac{\partial}{\partial y}(1) + x * \frac{\partial}{\partial y}(\ln y) = x^2 * 1 - x^3 z * 0 + x * \frac{1}{y} = x^2 + \frac{x}{y}$$

$$h_y(-2, 1, 3) = (-2)^2 + \frac{-2}{1} = 4 - 2 = 2$$

2. (b) For the unknown function $m(x, y)$, the *first partial derivative* $m_y = x \ln(x^2 - xy)$. Find m_{yx}.

$m_{yx} = \frac{\partial}{\partial x}(h_y) = \frac{\partial}{\partial x}\left[x \ln(x^2 - xy)\right]$, which requires use of the product rule.

Let $u = x$ and $v = \ln(x^2 - xy)$. Then $u_x = 1$ and $v_x = \frac{1}{x^2 - xy} * (2x - y) = \frac{2x - y}{x(x-y)}$.

Note that the chain rule was needed for v_x. There is ***no*** logarithm property which applies.

So $m_{yx} = u * v_x + u_x * v = x * \frac{2x - y}{x(x-y)} + 1 * \ln(x^2 - xy) = \frac{2x - y}{x - y} + \ln(x^2 - xy)$.

Note that the final answer is in simplest form—no further reduction of terms is possible.

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3. (a) Find the volume under the surface $z = 6x^2 y$ and above the rectangle $0 \le x \le 4,\ 0 \le y \le 3$.

$$\int_0^4 \int_0^3 6x^2 y \, dy \, dx = \int_0^4 \left[6x^2 * \frac{1}{2} y^2\right]_0^3 dx = \int_0^4 \left[3x^2 * 3^2 - 0\right] dx = \int_0^4 27x^2 \, dx = \left[9x^3\right]_0^4 = 9(4^3) - 0 = 576$$

alternate approach:

$$\int_0^3 \int_0^4 6x^2 y \, dx \, dy = \int_0^3 \left[2x^3 * y\right]_0^4 dy = \int_0^3 \left[2(4^3) * y - 0\right] dx = \int_0^3 128y \, dy = \left[64y^2\right]_0^3 = 64(3^2) - 0 = 576$$

3. (b) The rate of change of the population of Binthar, Montana, is given by $\frac{dy}{dt} = 0.02y$, where y is the population in thousands at time t, in years. Let $t = 0$ represent the year 2000, when the population was 300,000. Find the population function. Also state whether the population is growing or declining. (*Hint: If you recognize this one, you can write the function without solving the DE.*)

Exponential growth: $y = 300000 e^{0.02t}$; The population is rising. *The DE is solved by separation of variables.*

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4. (a) Use the first-order linear process to solve the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ with the initial value condition $y(0) = 3$.

$P(x) = \cos x \quad \Rightarrow \quad I(x) = e^{\int \cos x\, dx} = e^{\sin x}$

$e^{\sin x}\frac{dy}{dx} + \left(e^{\sin x}\cos x\right)y = e^{\sin x}\cos x \quad \Rightarrow \quad \int \frac{d}{dx}\left[e^{\sin x} y\right]dx = \int e^{\sin x}\cos x\, dx$

$u = \sin x \quad du = \cos x\, dx \Rightarrow \int e^{\sin x}\cos x\, dx = \int e^{u}\, du = e^{\sin x} + C$

$e^{\sin x} y = e^{\sin x} + C$

$y = e^{-\sin x}\left[e^{\sin x} + C\right] = 1 + Ce^{-\sin x}$

substitute initial condition values to find the value of C:

$3 = 1 + Ce^{-\sin 0} = 1 + Ce^{0} = 1 + C \quad \Rightarrow \quad 2 = C$, so the final answer is $y = 1 + 2e^{-\sin x}$.

4. (b) A researcher finds that the rate of productivity of worker bees in a newly-established hive is related to the size of the colony by the differential equation $\frac{dy}{dx} = y\left(x^2 + 1\right)$ with initial condition $y(0) = 2$. Solve to find an equation which represents the amount the worker bees produce as a function of colony size x.

method: separation of variables $\frac{1}{y}\frac{dy}{dx} = x^2 + 1 \quad \Rightarrow \quad \int \frac{1}{y}\, dy = \int x^2 + 1\, dx \quad \Rightarrow \quad \ln|y| = \frac{1}{3}x^3 + x + C$

$\Rightarrow \quad y = e^{\frac{1}{3}x^3 + x + C} = Ce^{\frac{1}{3}x^3 + x} \quad \Rightarrow \quad 2 = Ce^{\frac{1}{3}(0^3) + 0} \quad \Rightarrow \quad 2 = C \quad \Rightarrow \quad y = 2e^{\frac{1}{3}x^3 + x}$.

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5. (a) Let $f(t)$ be the solution to $y' = y^2 + 4t, \quad y(1) = 2$. Use Euler's method with $n = 2$ to estimate $f(2)$.

$h = \frac{2-1}{2} = \frac{1}{2}, \quad t_0 = 1, \quad y_0 = 2, \quad y' = g(1, 2) = 2^2 + 4(1) = 8$

$\Rightarrow \quad t_1 = \frac{3}{2}, \quad y_1 = 2 + 8\left(\frac{1}{2}\right) = 6, \quad y' = g\left(\frac{3}{2}, 6\right) = 6^2 + 4\left(\frac{3}{2}\right) = 42$

$\Rightarrow \quad t_2 = 2, \quad f(2) \cong y_2 = 6 + 42\left(\frac{1}{2}\right) = 27$

5. (b) Given the system of linear differential equations, $\frac{dx_1}{dt} = 3x_1 - 2x_2$ and $\frac{dx_2}{dt} = x_1 + e^{t}$, a student has found the necessary eigenvalues and eigenvectors to be $\lambda_1 = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_1 = 2 \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and has derived $\frac{dy_1}{dt} - y_1 = 2e^{t}$ and $\frac{dy_2}{dt} - 2y_2 = -e^{t}$. Finish the solution process to answer $x_1(t) = ?$ and $x_2(t) = ?$

See Exam 2 Sample 1 Solutions for the student's preliminary work.

$\frac{dy_1}{dt} - y_1 = 2e^{t} \quad \Rightarrow \quad I(t) = e^{\int -1\, dt} = e^{-t} \quad \Rightarrow \quad e^{-t}\frac{dy_1}{dt} - y_1 e^{-t} = 2e^{t} * e^{-t} \quad \Rightarrow \quad e^{-t} y_1 = 2t + C_1$

$\Rightarrow \quad y_1 = 2te^{t} + C_1 e^{t}$

$$\frac{dy_2}{dt} - 2y_2 = -e^t \Rightarrow I(t) = e^{\int -2\,dt} = -e^{-2t} \Rightarrow e^{-2t}\frac{dy_2}{dt} - 2y_2 e^{-2t} = -e^t * e^{-2t} \Rightarrow e^{-2t} y_2 = e^{-t} + C_2$$

$$\Rightarrow y_2 = e^t + C_2 e^{2t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = PY = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2te^t + C_1 e^t \\ e^t + C_2 e^{2t} \end{bmatrix} = \Rightarrow \begin{array}{c} x_1 = 2te^t + C_1 e^t + 2e^t + 2C_2 e^{2t} \quad or \quad e^t(2t + C_1 + 2) + 2C_2 e^{2t} \\ x_2 = 2te^t + C_1 e^t + e^t + C_2 e^{2t} \quad or \quad e^t(2t + C_1 + 1) + C_2 e^{2t} \end{array}$$

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