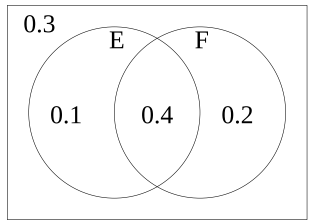
Math 131 Exam 3 Sample 2 Solutions

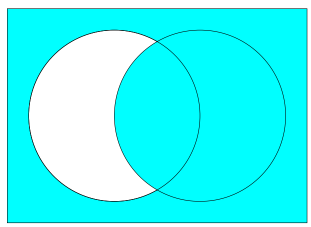
1. (a) i. The Venn diagram is:



ii. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.4}{0.6} = \frac{2}{3}$

iii. No, E and F are not mutually exclusive because $E \cap F \neq \emptyset$.

iv. $E' \cup F$ is shown here:



(b) The test manufacturer would be most interested in $P(B|A)$ because they are most interested in the efficacy of their test for someone who actually has the disease.
Note: The *efficacy* of something is how well it does what it is designed to do.

2. (a) Define events:
E is the event where the first patient has condition A.
F is the event where the second patient has condition C.
We are interested in $P(E \cap F')$. We use:

$$P(E \cap F') = P(F'|E)P(E) = \left(\frac{69}{99}\right)\left(\frac{20}{100}\right) = \frac{23}{165}$$

Note: It's also fine to define F to be the event where the second patient does not have condition C. In that case we'd be interested in $P(E \cap F)$. The answer will be the same.

(b) The information given is:
$P(E) = 0.002$
$P(F|E) = 0.93$
$P(F|E') = 0.05$ We are interested in $P(E|F)$ and so we use Bayes' Theorem with the E and F interchanged:

$$\begin{aligned} P(E|F) &= \frac{P(E)P(F|E)}{P(E)P(F|E) + P(E')P(F|E')} \\ &= \frac{(0.002)(0.93)}{(0.002)(0.93) + (0.998)(0.05)} \\ &= \frac{0.00186}{0.05176} = \frac{31}{2598} \approx 0.0359 \end{aligned}$$

3. (a) The PDF is $f(x) = 0.2e^{-0.2x}$.

   (b) Note: Living to 75 or above is 5 additional years. Also I've done two decimal places rather than one.
   This probability would be

$$\begin{aligned}
\int_5^\infty 0.2e^{-0.2x}\ dx &= \lim_{b\to\infty} \int_5^b 0.2e^{-0.2x}\ dx \\
&= \lim_{b\to\infty}\ -e^{-0.2x}\Big|_5^b \\
&= \lim_{b\to\infty}\ -\frac{1}{e^{0.2b}} + \frac{1}{e^{0.2(5)}} \\
&= \frac{1}{e} \\
&\approx 0.37
\end{aligned}$$

   (c) Now we want to find the upper limit Y so that 90% of all 70-year olds will live under that many years. In other words:

$$\begin{aligned}
\int_0^Y 0.2e^{-0.2x}\ dx &= 0.9 \\
-e^{-0.2x}\Big|_0^Y &= 0.9 \\
-e^{-0.2Y} + 1 &= 0.9 \\
e^{-0.2Y} &= 0.1 \\
-0.2Y &= \ln(0.1) \\
Y &= \frac{\ln(0.1)}{-0.2} \\
Y &\approx 11.5
\end{aligned}$$

   Thus 90% of 70-year olds will live 11.5 or fewer additional years.

4. (a) A single organism has mean 2 and standard deviation 1. Thus by the CLT a sample of 50 will have approximately normal distribution with $\mu = 2$ and $\sigma = 1/\sqrt{50}$.

   We want $P(1.7 \leq X \leq 2.1)$. We convert to z-scores to get:

$$P\left(\frac{1.7-2}{1/\sqrt{50}} \leq Z \leq \frac{2.1-2}{1/\sqrt{50}}\right) = P(-2.12 \leq Z \leq 0.71) = 0.7611 - 0.0170 = 0.7441$$

   (b) For a sample of 100 organisms $\mu = 2$ and $\sigma = 1/\sqrt{100} = 0.1$. Now be careful. We want the area *above* our z-score to be 10% which means the area below is 90%. The closest we can get is using $z = 1.28$ so then:

$$\begin{aligned} 1.28 &= \frac{x-2}{0.2} \\ x - 2 &= 2.56 \\ x &= 4.56 \end{aligned}$$

   So only 10% of our organisms will live over 4.56 years.

5. (a) i. We need

$$\begin{aligned} \int_0^5 kx^2 \; dx &= 1 \\ \frac{k}{3}x^3\Big|_0^5 &= 1 \\ \frac{k}{3}(125) - \frac{k}{3}(0) &= 1 \\ k &= \frac{3}{125} \end{aligned}$$

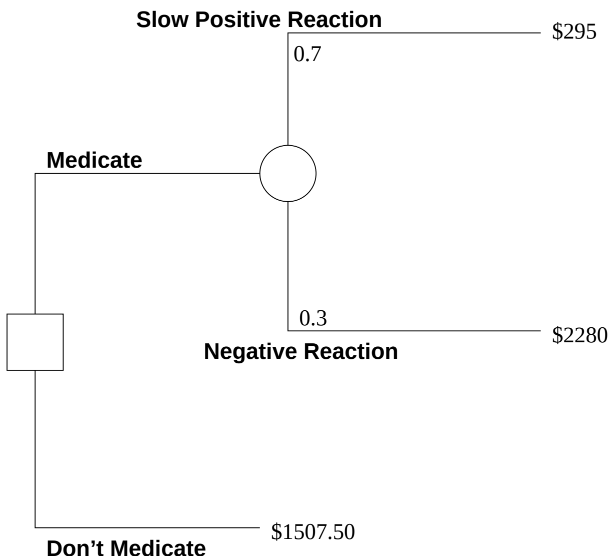
   ii. Then

$$E(X) = \int_0^5 x\left(\frac{3}{125}x^2\right)\; dx = \int_0^5 \frac{3}{125}x^3 \; dx = \frac{3}{500}x^4\Big|_0^5 = \frac{3}{500}(625) = 3.75$$

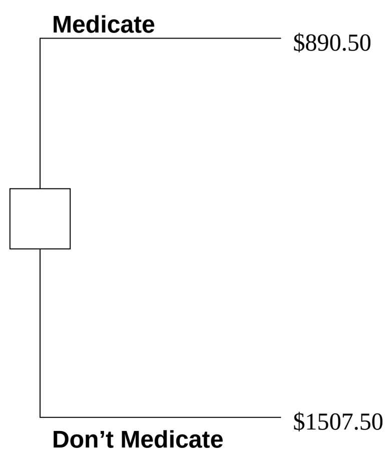
   iii. And

$$\begin{aligned} Var(X) &= \int_0^5 x^2\left(\frac{3}{125}x^2\right)\; dx - (3.75)^2 \\ &= \int_0^5 \frac{3}{125}x^4 \; dx - (3.75)^2 \\ &= \frac{3}{625}x^5\Big|_0^5 - (3.75)^2 \\ &= \frac{3}{500}(3125) - (3.75)^2 \\ &= 4.6875 \end{aligned}$$

(b) Here is the first pruning:



Second pruning:



Thus the best choice is to medicate at just over half the cost on average.