Calculus 131, Final Exam sample 1 key

Fall 2010

1(a) Suppose nutritional supplement I contains 5 mg of niacin and 2 mg of fish oil while nutritional supplement II contains 2 mg of niacin and 6 mg of fish oil. You take x units of I and y units of 2. Write down the system of equations which would correspond to needing A mg of niacin and B mg of fish oil, where A and B are unknown constants. Solve using the inverse of a matrix.

equations: niacin $5x + 2y = A$ *fish oil* $2x + 6y = B$

in matrix form: $\begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} A \\ B \end{bmatrix}$

to find the needed inverse, either use the "shortcut formula", or -

$$\left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{2}{5} & \frac{1}{5} & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{26}{5} & -\frac{2}{5} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{1}{13} & \frac{5}{26} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{13} & -\frac{1}{13} \\ 0 & 1 & -\frac{1}{13} & \frac{5}{26} \end{array}\right]$$

So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & \frac{5}{25} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow x = \frac{3}{13}A - \frac{1}{13}B,\ y = -\frac{1}{13}A + \frac{5}{26}B$

1(b) Suppose the Leslie matrix for a population of juveniles and adults is shown below.

interpretation: "juveniles and adults" defines x = number of juveniles and y = number of adults

$$\begin{bmatrix} 0.5 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}$$

If there are 500 adults in a stable population then how many juveniles are there?

method: eigenvalues and eigenvectors.

$$M - \lambda I = \begin{bmatrix} 0.5 & -0.1 \\ 0.2 & 0.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0.5-\lambda & -0.1 \\ 0.2 & 0.2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(M - \lambda I) &= (0.5-\lambda)(0.2-\lambda) - (0.2)(-0.1) = 0 \\ &= 0.1 - 0.7\lambda + \lambda^2 + 0.02 = 0 \\ &= \lambda^2 - 0.7\lambda - 0.12 = 0 \\ &= 100\lambda^2 - 70\lambda + 12 = 0 \\ &= (10\lambda - 3)(10\lambda - 4) = 0 \\ &\Rightarrow \quad \lambda = 0.3 \quad \lambda = 0.4 \end{aligned}$$

For $\lambda = 0.3$

$$M - 0.3I = \begin{bmatrix} 0.5-0.3 & -0.1 \\ 0.2 & 0.2-0.3 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.1 \\ 0.2 & -0.1 \end{bmatrix} \to \begin{bmatrix} 0.2 & -0.1 \\ 0 & 0 \end{bmatrix}$$

$$0.2x - 0.1y = 0 \quad \Rightarrow \quad 0.2x = 0.1y \quad \Rightarrow \quad x = \frac{1}{2}y$$

"500 adults" is $y = 500$, so number of juveniles $x = 250$.

side note: The factor of 0.3 means the populations (juveniles and adults) is decreasing by 30% per time period.

For $\lambda = 0.4$

$$M - 0.4I = \begin{bmatrix} 0.5-0.4 & -0.1 \\ 0.2 & 0.2-0.4 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.1 \\ 0.2 & -0.2 \end{bmatrix} \to \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0 \end{bmatrix}$$

$0.1x - 0.1y = 0 \quad \Rightarrow \quad 0.1x = 0.1y \quad \Rightarrow \quad x = y$

"500 adults" is $y = 500$, so number of juveniles $y = 500$.

side note: The factor of 0.4 means the populations (juveniles and adults) is decreasing by 40% per time period.

2(a) The calorie requirement for a certain athlete is $C(x, y) = 4xy^2 + e^{0.01xy}$ where x is the athlete's age in years and y is the number of hours she exercises daily.

i. Find, simplify and interpret $C(16, 6)$.

$C(16, 6) = 4(16)(6^2) + e^{0.01(16)(6)} = 2304 + e^{0.96} \approx 2{,}306.61$; An athlete who is 16 years old and exercises 6 hours daily requires about 2306.61 calories.

ii. Find, simplify and interpret $C_x(20, 4)$.

$C_x = 4y^2 + 0.01ye^{0.01xy} \quad \Rightarrow \quad C_x(20, 4) = 4(4^2) + 0.01(4)e^{0.01(20)(4)} = 64 + e^{0.8} \approx 66.23$; An athlete who continues to exercise 4 hours daily will require about 66.23 more calories than before as she ages from 20 to 21 years.

iii. Suppose you set $C_y(x, 18) = 500$ and solved for x. Without actually doing this (don't!), explain what you would be finding.

An athlete who is x years old is increasing her daily exercise from 18 to 19 hours daily, and as a result is increasing her calorie requirement by about 500 more calories.

2(b) Find and categorize all relative maxima, minima and saddle points for $f(x, y) = 3x^2 - 6xy + y^3 - 9y$.

$f_x = 6x - 6y = 0 \qquad f_y = -6x + 3y^2 - 9 = 0 \qquad$ Solve f_x for $y = \ldots$ and use substitution.

$-6y = -6x \quad \Rightarrow \quad y = x \quad \Rightarrow \quad -6y + 3y^2 - 9 = 0 \quad \Rightarrow \quad 3y^2 - 6y - 9 = 3(y^2 - 2y - 3) = 3(y+1)(y-3) = 0$

$y = -1 \quad \Rightarrow \quad x = -1, \qquad y = 3 \quad \Rightarrow \quad x = 3$

When $y = -1$, $x = -1$, $f(-1, -1) = 3(-1)^2 - 6(-1)(-1) + (-1)^3 - 9(-1) = 5$, first point is $(-1, -1, 5)$.

When $y = 3$, $x = 3$, $f(3, 3) = 3(3)^2 - 6(3)(3) + (3)^3 - 9(3) = -27$, second point is $(3, 3, -27)$.

$f_{xx} = \frac{\partial}{\partial x}(6x - 6y) = 6 \qquad f_{yy} = \frac{\partial}{\partial y}(-6x + 3y^2 - 9) = 6y \qquad f_{xy} = \frac{\partial}{\partial y}(6x - 6y) = -6$

$D(x, y) = 6(6y) - (-6)^2 = 36y - 36$

At $(-1, -1, 5)$: $D(-1, -1) = 36(-1) - 36 = -72 < 0 \quad \Rightarrow \quad$ saddle point.

At $(3, 3, -27)$: $D(3, 3) = 36(3) - 36 = 72 > 0$, $f_{xx} = 6 > 0 \quad \Rightarrow \quad$ relative minimum.

3(a) Show that the eigen-pairs for $M = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ are $\lambda = 2$ with $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are $\lambda = 1$ with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

work to find eigenvalues and eigenvectors:

$$M = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \qquad M - \lambda I = \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix}$$

$$\det(M - \lambda I) = (4-\lambda)(-1-\lambda) - (2)(-3) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = 2, \ \lambda = 1$$

$$\begin{bmatrix} 4-2 & -3 \\ 2 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4-1 & -3 \\ 2 & -1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3(b) Solve the following system of differential equations: $\frac{dx_1}{dt} = 4x_1 - 3x_2$, $\frac{dx_2}{dt} = 2x_1 - x_2 + 1$

in matrix form: $X' = MX + Q$, so $M = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

using 3(a), identify D, P and P^{-1}*:*

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & 3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

solve for y_1 *and* y_2*:*

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = DY + P^{-1}Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} \frac{dy_1}{dt} = 2y_1 - 1 \\ \frac{dy_2}{dt} = y_2 + 3 \end{array}$$

$$\frac{dy_1}{dt} - 2y_1 = -1 \Rightarrow I(t) = e^{\int -2\,dt} = e^{-2t} \Rightarrow e^{-2t}y_1 = \int -e^{-2t}\,dt$$

$$\Rightarrow e^{-2t}y_1 = \tfrac{1}{2}e^{-2t} + C_1 \Rightarrow y_1 = \tfrac{1}{2} + C_1e^{2t}$$

$$\frac{dy_2}{dt} - y_2 = 3 \Rightarrow I(t) = e^{\int -1\,dt} = e^{-t} \Rightarrow e^{-t}y_2 = \int 3e^{-t}\,dt$$

$$\Rightarrow e^{-t}y_2 = -3e^{-t} + C_2 \Rightarrow y_2 = -3 + C_2e^{t}$$

solve for x_1 *and* x_2*:*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = PY = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} + C_1e^{2t} \\ -3 + C_2e^{t} \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x_1 = \frac{3}{2} + 3C_1e^{2t} - 3 + C_2e^{t} = -\frac{3}{2} + 3C_1e^{2t} + C_2e^{t} \\ x_2 = 1 + 2C_1e^{2t} - 3 + C_2e^{t} = -2 + 2C_1e^{2t} + C_2e^{t} \end{array}$$

check (not necessary for your exam):

$$x_1 = -\tfrac{3}{2} + 3C_1e^{2t} + C_2e^t \quad \Rightarrow \quad \frac{dx_1}{dt} = 6C_1e^{2t} + C_2e^t$$

$$4x_1 - 3x_2 = -6 + 12C_1e^{2t} + 4C_2e^t + 6 - 6C_1e^{2t} - 3C_2e^t = (-6+6) + \left(12C_1e^{2t} - 6C_1e^{2t}\right) + \left(4C_2e^t - 3C_2e^t\right)$$

$$x_2 = -2 + 2C_1e^{2t} + C_2e^t \quad \Rightarrow \quad \frac{dx_2}{dt} = 4C_1e^{2t} + C_2e^t$$

$$2x_1 - x_2 + 1 = -3 + 6C_1e^{2t} + 2C_2e^t + 2 - 2C_1e^{2t} - C_2e^t + 1 = (-3+2+1) + \left(6C_1e^{2t} - 2C_1e^{2t}\right) + \left(2C_2e^t - C_2e^t\right)$$

4(a) Suppose the rate of growth on day t of a certain population is given by $r(t) = t \ln t$. Use the Trapezoidal Rule with $n = 4$ to approximate the change in population between $t = 1$ and $t = 3$.

change in population = area under the curve $\approx \frac{b-a}{n} * \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right]$

change in population $\approx \frac{3-1}{4} * \left[\frac{1}{2}f(1) + f(1.5) + f(2) + f(2.5) + \frac{1}{2}f(3)\right]$

change in population $\approx \frac{1}{2} * \left[\frac{1}{2}(1)(\ln 1) + (1.5)(\ln 1.5) + (2)(\ln 2) + (2.5)(\ln 2.5) + \frac{1}{2}(3)(\ln 3)\right]$

change in population ≈ 2.967

4(b) Suppose an operation has a 40% chance of costing \$1000 and a 60% chance of costing \$2000.

i. Suppose X is the random variable for the cost of one operation. Find $E(X)$ and $\sigma(X)$.

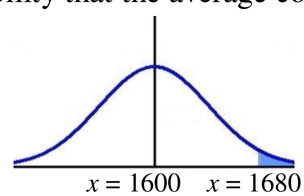
$$\mathrm{E}(X) = 1000(0.4) + 2000(0.6) = 400 + 1200 = 1600$$

$$\mathrm{Var}(X) = (1000-1600)^2(0.4) + (2000-1600)^2(0.6) = 360000(0.4) + 160000(0.6) = 240000$$

$$\sigma(X) = \sqrt{240000} = 200\sqrt{6}$$

ii. Suppose 100 operations take place. Use the CLT to approximate the probability that the average cost will be \$1680 or more.

$$z = \frac{1680-1600}{200\sqrt{6}/\sqrt{100}} = \frac{1680-1600}{20\sqrt{6}} \approx 1.63$$



from the table $P(-\infty < z < 1.63) = 0.9484$

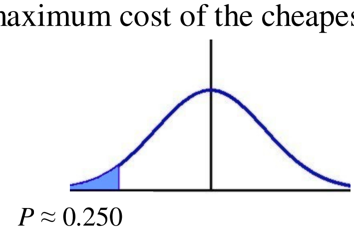
$P(x > 1680) = P(1.63 < z < \infty) = 1 - P(-\infty < z < 1.63) = 1 - 0.9484 = 0.0516$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	**0.9484**	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633

iii. Suppose 100 operations take place. Use the CLT to approximate the maximum cost of the cheapest 25% of the operations.

from the table $P(-\infty < z < -0.67) \approx 0.2500$

$$-0.67 = \frac{x-1600}{20\sqrt{6}} \quad \Rightarrow \quad x = -0.67\left(20\sqrt{6}\right) + 1600 \approx \$1567.18$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	**0.2514**	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776

5(a) Suppose the lifespan of a chimpanzee is exponentially distributed with average 20 years.

i. What is the probability that a chimpanzee will live over 15 years?

For $f(x) = ae^{-ax}$, $\mu = \frac{1}{a}$, so $\mu = 20 \Rightarrow f(x) = \frac{1}{20}e^{-\frac{1}{20}x} = 0.05e^{-0.05x}$

$$\int_{15}^{\infty} 0.05e^{-0.05x}\,dx = \lim_{b\to\infty}\left[\frac{0.05}{-0.05}e^{-0.05x}\right]_{15}^{b} = \lim_{b\to\infty}\left[-e^{-0.05b} - \left(-e^{-0.05(15)}\right)\right] = e^{-0.75} \approx 0.4724$$

You also could have answered this question by calculating $1 - \int_0^{15} 0.05e^{-0.05x}\,dx$.

ii. Over how many years will just 5% of chimpanzees live?

$$\int_{a}^{\infty} 0.05e^{-0.05x}\,dx = \lim_{b\to\infty}\left[\frac{0.05}{-0.05}e^{-0.05x}\right]_{a}^{b} = \lim_{b\to\infty}\left[-e^{-0.05b} - \left(-e^{-0.05(a)}\right)\right] = e^{-0.05(a)} \approx 0.0500$$

$$\Rightarrow \ln\left[e^{-0.05(a)}\right] = -0.05a \approx \ln(0.05) \Rightarrow a \approx \frac{\ln(0.05)}{-0.05} \approx 59.9$$

You also could have answered this question by just bringing down the result of the previous part, without the need to re-work the integration.

5(b) A bag contains 5 red balls and 10 yellow balls. You remove two without replacement.

i. What is the probability that the first is red and the second is red?

$$\frac{5}{15} * \frac{4}{14} = \frac{1}{3} * \frac{2}{7} = \frac{2}{21}$$

ii. What is the probability that the first is not a red and the second is red?

$$\frac{10}{15} * \frac{5}{14} = \frac{2}{3} * \frac{5}{14} = \frac{5}{21}$$

iii. What is the probability that the second is red?

$$\frac{2}{21} + \frac{5}{21} = \frac{7}{21}$$

6(a) Suppose a discrete population has iterative function $f(x) = 2.5x(1 - x)$.

i. Find the fixed point.

$$2.5x - 2.5x^2 = x \Rightarrow 1.5x - 2.5x^2 = 5x(0.3 - 0.5x) \Rightarrow x = \frac{0.3}{0.5} = \frac{3}{5} = 0.6$$

ii. Determine using the derivative whether the fixed point is stable or not.

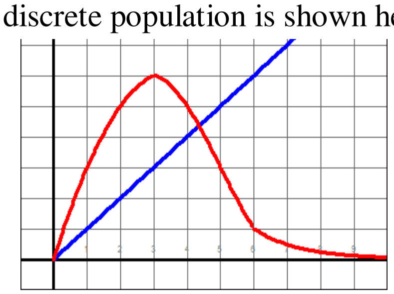
$$f' = 2.5 - 5x \Rightarrow f'(0.6) = 2.5 - 5(0.6) = 2.5 - 3 = -0.5 \Rightarrow |f'(0.6)| < 1$$

This fixed point is stable.

iii. Suppose $a_1 = 0.5$. Find a_2 through a_5. Approximate each to two decimal digits.

$a_2 = 2.5(0.5)(1-0.5) \approx 0.63$, $a_3 = 2.5(0.63)(1-0.63) \approx 0.58$, $a_4 = 2.5(0.58)(1-0.58) \approx 0.61$, $a_5 = 2.5(0.61)(1-0.61) \approx 0$

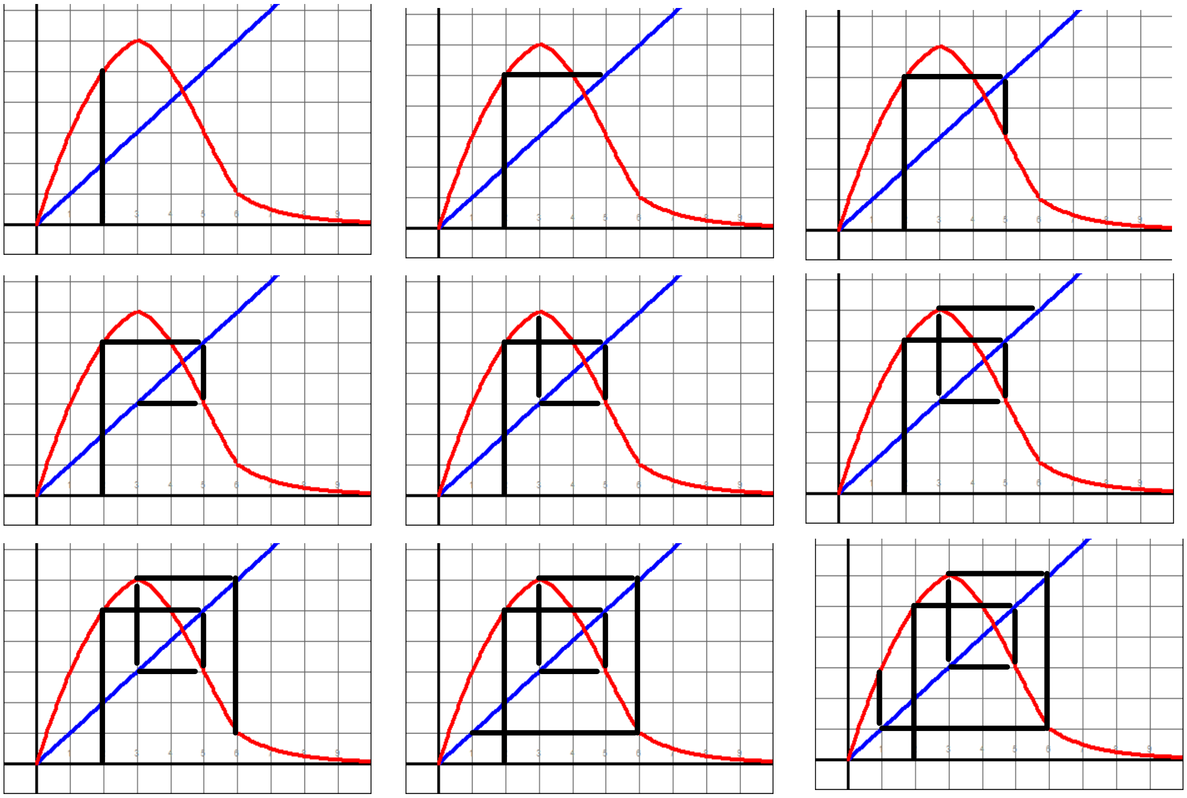
6(a) Suppose the iterative function for a discrete population is shown here. The grid is at 1 units.

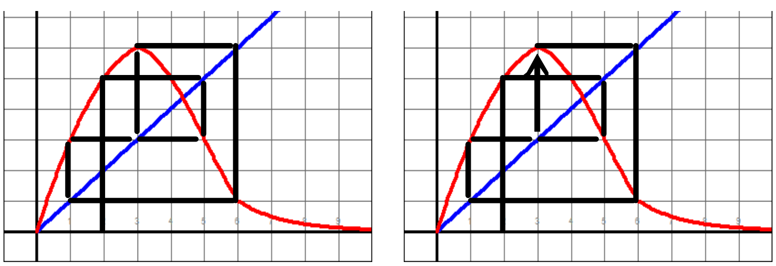


i. What is the approximate fixed point? *estimate the intersection: x = 4.2 or 4.3*

ii. Assuming that $a_1 = 2$, cobweb until you get into a never-ending cycle which repeats three values. All should be exact! Write down your values (up to those three) and also recopy your cobweb diagram as well as you can onto the answer sheet.

On your exam, you'd only need the next-to-last picture below. I included all of them to show the process.





The values begin cycling at the up arrow.

$a_1 = 2$, $a_2 = 5$, $a_3 = 3$, $a_4 = 6$, $a_5 = 1$, $a_6 = 3$, $a_7 = 6$, $a_8 = 1$, $a_9 = 3$, etc.

iii. If a_n represents the number of rabbits in thousands during year n, explain in words what is happening to the rabbit population in the long term.

In year 3, there are 3000 rabbits. In year 4, the number of rabbits grows to 6000, then in year 5 the number diminishes to 1000. In year 6 the cycle starts repeating 3000, 6000, 1000, 3000, 6000, 1000, etc.

iv. Suppose a_1 is very large. Explain what you can say about the next few iterations.

If a_1 is very large, then a_2 will be quite small. Then the next few iterations (a_3, a_4 and a_5) will show gradual growth in the population, until population size "tops out" at 6000.