Math 131 Final Exam Sample 2 Solution **Fall 2010**

1. (a) i. Supplement I contains 20mg of vitamin A per supplement.

ii. Bill took 5 of supplement II.

iii. If the first is $X = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and the second is $Y = \begin{bmatrix} 20 & 15 \\ 10 & 30 \end{bmatrix}$ then X is supp×Bill and Y is vitamin×supp and to make the multiplication work we need YX which will then be vitamin×Bill:

$$YX = \begin{bmatrix} 20 & 15 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 135 \\ 180 \end{bmatrix} \longrightarrow$$

# mg	Bill
A	135
B	180

This matrix shows the amount of each vitamin that Bill took.

(b) i. After one iteration:

$$\begin{bmatrix} 0.2 & 0.7 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \end{bmatrix} = \begin{bmatrix} 620 \\ 660 \end{bmatrix}$$

and after two:

$$\begin{bmatrix} 0.2 & 0.7 \\ 0.3 & 0.6 \end{bmatrix} = \begin{bmatrix} 620 \\ 660 \end{bmatrix} = \begin{bmatrix} 586 \\ 582 \end{bmatrix}$$

ii. The eigenvalues are the solutions of

$$0 = \det \begin{bmatrix} 0.2 - \lambda & 0.7 \\ 0.3 & 0.6 - \lambda \end{bmatrix} = ... = (\lambda - 0.9)(\lambda + 0.1)$$

We choose the positive $\lambda = 0.9$. Then we solve

$$\begin{bmatrix} 0.2 - 0.9 & 0.7 & 0 \\ 0.3 & 0.6 - 0.9 & 0 \end{bmatrix} \to \begin{bmatrix} -0.7 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} x \\ y \end{bmatrix}$ is the eigenvector with $-0.7x + 0.7y = 0$ or $x = y$. If there are 2000 all together then there are 1000 of each.

2. (a) We have

$$\begin{aligned}
\int_0^2 \int_0^{x+1} x+y \ dy \ dx &= \int_0^2 xy + \frac{1}{2}y^2 \Big|_0^{x+1} \ dx \\
&= \int_0^2 \left[x(x+1) + \frac{1}{2}(x+1)^2\right] - \left[x(0) + \frac{1}{2}(0)^2\right] \ dx \\
&= \int_0^2 \frac{3}{2}x^2 + 2x + \frac{1}{2} \ dx \\
&= \frac{1}{2}x^3 + x^2 + \frac{1}{2}x \Big|_0^2 \\
&= \left[\frac{1}{2}(2)^3 + (2)^2 + \frac{1}{2}(2)\right] - \left[\frac{1}{2}(0)^3 + (0)^2 + \frac{1}{2}(0)\right]
\end{aligned}$$

(b) We have $f_x = 3x^2 - 12x = 0$ so $3x(x-4) = 0$ so $x = 0$ or $x = 4$.
We have $f_y = 6 - 2y = 0$ so $y = 3$.
Thus our two potential points are $(0,3)$ and $(4,3)$.
Then $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (6x-12)(-2) - (0)^2 = -12x + 24$.
$(0,3)$: $D(0,3) = +$ and $f_{xx}(0,3) = -$ so $(0,3)$ is a relative minimum.
$(4,3)$: $D(4,3) = -$ so $(4,3)$ is a saddle point.

3. (a) The initial value problem would be $\frac{dy}{dt} = 0.032y + 200$ with $y(0) = 1000$.

(b) This is separable:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2\sqrt{y}}{x} + \sqrt{y} \\
\frac{dy}{dx} &= \sqrt{y}\left(\frac{2}{x} + 1\right) \\
y^{-1/2}\ dy &= \frac{2}{x} + 1\ dx \\
\int y^{-1/2}\ dy &= \int \frac{2}{x} + 1\ dx \\
2\sqrt{y} &= 2\ln|x| + x + C \\
\sqrt{y} &= \ln|x| + \frac{1}{2}x + \frac{C}{2} \\
y &= \left(\ln|x| + \frac{1}{2}x + \frac{C}{2}\right)^2
\end{aligned}$$

(c) We should assume $t > 0$. This should have been given. Then this is first-order linear:

$$\begin{aligned}
t\frac{dy}{dt} &= t^3 - t - y \\
\frac{dy}{dt} &= t^2 - 1 - \left(\frac{1}{t}\right)y \\
\frac{dy}{dt} + \left(\frac{1}{t}\right)y &= t^2 - 1
\end{aligned}$$

We have $P(t) = \frac{1}{t}$ so $S(t) = \ln|t| = \ln t$. Thus the solution is as follows:

$$\begin{aligned}
y &= e^{-\ln t}\int (t^2 - 1)e^{\ln t}\ dt \\
&= \frac{1}{t}\int t^3 - t\ dt \\
&= \frac{1}{t}\left[\frac{1}{4}t^4 - \frac{1}{2}t^2 + C\right] \\
&= \frac{1}{4}t^3 - \frac{1}{2}t + \frac{C}{t}
\end{aligned}$$

4. (a) The table is:

x	$p(x)$	
0	$1/8 = 0.125$	(Corresponding to TTT.)
1	$3/8 = 0.375$	(Corresponding to TTH, THT and HTT.)
2	$3/8 = 0.375$	(Corresponding to THH, HTH and HHT.)
3	$1/8 = 0.125$	(Corresponding to HHH.)

(b) $P(X \leq 2) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$.

(c) We have

$$E(X) = 0.125(0) + 0.375(1) + 0.375(2) + 0.125(3) = 1.5$$

and

$$\begin{aligned}\sigma(X) &= \sqrt{Var(X)} \\ &= \sqrt{0.125(0-1.5)^2 + 0.375(1-1.5)^2 + 0.375(2-1.5)^2 + 0.125(3-1.5)^2} \\ &= \sqrt{0.75}\end{aligned}$$

(d) With the CLT and 100 times we use $\mu = 1.5$ and $\sigma = \sqrt{0.75}/\sqrt{100} = \sqrt{0.75}/10$. We convert to z:

$$z = \frac{1.4 - 1.5}{\sqrt{0.75}/10} = -1.15 \qquad \text{and} \qquad z = \frac{1.65 - 1.5}{\sqrt{0.75}/10} = 1.73$$

and so

$$P(1.4 \leq X \leq 1.65) = P(-1.15 \leq Z \leq 1.73) = 0.9582 - 0.1251 = 0.8331$$

(e) If there is a 0.05 probability above then there is a 0.95 probability below so we look up 0.95 within the table and get $z = 1.64$ (or $z = 1.65$, they're equally close - you could be sneaky and use $z = 1.645$ but we won't) and then:

$$\begin{aligned}1.64 &= \frac{x - 1.5}{\sqrt{0.75}/10} \\ x &= 1.6420\end{aligned}$$

So there is only a 5% probability that the average will lie above 1.6420.

5. (a) We have $f(x) = ae^{-ax}$ and:

$$\begin{aligned}
\int_6^{\infty} ae^{-ax}\ dx &= 0.2 \\
\lim_{b\to\infty}\ -e^{-ax}\Big|_6^b &= 0.2 \\
\lim_{b\to\infty}\ -\frac{1}{e^{a(b)}} + \frac{1}{e^{a(6)}} &= 0.2 \\
e^{-6a} &= 0.2 \\
-6a &= \ln(0.2) \\
a &= -\frac{1}{6}\ln(0.2)
\end{aligned}$$

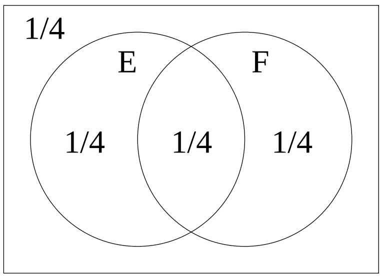
so the mean is

$$\mu = \frac{1}{a} = \frac{1}{-\frac{1}{6}\ln(0.2)} \approx 3.73 \text{ minutes}$$

(b) We wish to know $P(E|F)$. Bayes' Theorem tells us:

$$\begin{aligned}
P(E|F) &= \frac{P(E)P(F|E)}{P(E)P(F|E) + P(E')P(F|E')} \\
&= \frac{(2/6)(1/5)}{(2/6)(1/5) + (4/6)(2/5)}
\end{aligned}$$

(c) The key is that there must be overlap between E and F but $P(E|F)$ must equal $P(E)$. Here's an example:



Note: You might think this is tricky but there's a trick to the tricky trickiness. What I did was just think of two events which satisfy that criteria and then use those as my model. I thought about flipping two coins and letting E be a head on the first coin and F be a head on the second coin. We know for a fact these are independent but not mutually exclusive. My Venn diagram corresponds to this real world situation.

6. (a) i. The fixed point is located where $\frac{1}{16}x(x-1)^2 = x$. If $x = 0$ we get the trivial fixed point and if $x \neq 0$ we cancel it to get

$$\begin{aligned}
\frac{1}{16}(x-1)^2 &= 1 \\
(x-1)^2 &= 16 \\
x - 1 &= \pm 4 \\
x &= 1 \pm 4
\end{aligned}$$

The only one which makes sense as a population is $x = 5$.

ii. We have

$$\begin{aligned}
a_1 &= 5.1 \\
a_2 &= f(5.1) = 5.4 \\
a_3 &= f(5.4) = 6.5 \\
a_4 &= f(6.5) = 12.3 \\
a_5 &= f(12.3) = 98.2
\end{aligned}$$

iii. Not stable because even though we started close to the fixed point 5 we moved away from it.

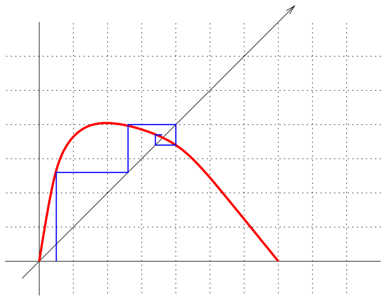
iv. We have

$$\begin{aligned}
f(x) &= \frac{1}{16}(x^3 - 2x^2 + x) \\
f'(x) &= \frac{1}{16}(3x^2 - 4x + 1) \\
|f'(5)| &= \frac{1}{16}(3(25) - 4(4) + 1) \\
|f'(5)| &= \frac{1}{16}(60) \\
|f'(5)| &> 1
\end{aligned}$$

Here again we see it's unstable.

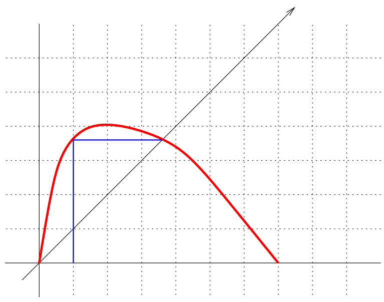
(b) i. The approximate fixed point is $x \approx 3.7$.

ii. This is how it looks, more or less:



Here we have, approximately:
$a_1 = 0.5$
$a_2 = 2.6$
$a_3 = 4$
$a_4 = 3.3$
$a_5 = 3.7$

iii. We see $a_1 = 1$ does the job, as shown by this cobweb. Note that the cobweb is not mandatory nor requested.



iv. The largest that a_1 could be is 7 because there is no function beyond that point on the graph.

v. The largest that a_2 could be is about 4 because that's the highest the function goes (remember how we get a_2).