Function Composition Examples/Notes (prepared by Tim Pilachowski)

For all examples, let $f(x) = 2x + 3$, $g(x) = \sqrt{x} - 1$, and $h(x) = \sqrt{x-1}$. Note that *g* and *h* are not the same! One way to combine functions is composition: inserting one function “into” the other. Composition can be written either as “$(f \circ g)(x)$” or “ $f \circ g$”. These are read as “*f* composed with *g*” or “ *f* of $g(x)$”. The compositions $(f \circ g)(x)$, $g \circ f$, and $h \circ f$ are evaluated as follows:

$$\begin{aligned}(f \circ g)(x) &= f(g[x]) \\ &= f(\sqrt{x} - 1) \\ &= 2(\sqrt{x} - 1) + 3 \\ &= 2\sqrt{x} - 2 + 3 \\ &= 2\sqrt{x} + 1\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f[x]) \\ &= g(2x + 3) \\ &= \sqrt{2x + 3} - 1\end{aligned}$$

$$\begin{aligned}(h \circ f)(x) &= h(f[x]) \\ &= h(2x + 3) \\ &= \sqrt{(2x + 3) - 1} \\ &= \sqrt{2x + 2}\end{aligned}$$

Domains and ranges

The domains and ranges of compositions of functions may differ from those of the original functions.

Function	**Domain**	**Range**
f	$(-\infty, \infty)$	$(-\infty, \infty)$
g	$[0, \infty)$	$[-1, \infty)$
$f \circ g$	$[0, \infty)$	$[1, \infty)$
$g \circ f$	$\left[-\frac{3}{2}, \infty\right)$	$[-1, \infty)$

Using a graphing calculator

You can use a TI-82 or TI-83 to graph and evaluate compositions of functions. For these examples $f(x) = 2x + 3$ will be entered as Y1 $= 2x + 3$ and $g(x) = \sqrt{x} - 1$ will be entered as Y2 $= \sqrt{\ }(x) - 1$.

The compositions can be entered in terms of Y1 and Y2 using the calculator’s Y-VARS keys:

Combination	Calculator entry	Combination	Calculator entry
$f \circ g$	Y3 = Y1 (Y2)	$g \circ f$	Y3 = Y2 (Y1)

Evaluating compositions at specific points

When composing functions and evaluating at a specific value of *x*, for example $(f \circ g)(4)$ and $(g \circ f)(4)$, you will get the same result whether you (A) compose first to get the algebraic rule, then evaluate, or (B) evaluate the functions first, then compose them. Both approaches are illustrated side-by-side below.

Evaluation	(A) compose first, then evaluate	(B) evaluate first, then compose
$(f \circ g)(4)$	$(f \circ g)(x) = 2\sqrt{x} + 1$ $(f \circ g)(4) = 2\sqrt{4} + 1 = 2(2) + 1$ $= 5$	$g(4) = \sqrt{4} - 1 = 2 - 1 = 1$ $(f \circ g)(4) = f(g(4)) = f(1)$ $= 2(1) + 3$ $= 5$
$(g \circ f)(4)$	$(g \circ f)(x) = g(2x + 3)$ $(g \circ f)(4) = \sqrt{2(4) + 3} - 1$ $= \sqrt{11} - 1$	$f(4) = 2(4) + 3 = 11$ $(g \circ f)(4) = g(f(4)) = g(11)$ $= \sqrt{11} - 1$

Composition and inverse functions

If you think of a function as being like a machine, then you can take an *x*-value into your machine and get the *y*-value out of it:

$$x \longrightarrow f(x) \longrightarrow y$$

The inverse function for *f*, labeled f^{-1} and called "f inverse" takes you from the *y*-value back to the *x*-value:

$$y \longrightarrow f^{-1}(y) \longrightarrow x$$

Given a function f and its inverse, f^{-1}, the following will always be true:

1. If $f(a) = b$, then $f^{-1}(b) = a$.
2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}.
3. The domain of *f* = the range of f^{-1}, and the range of *f* = the domain of f^{-1}.

Determining whether or not a function has an inverse

You can tell from the graph of a function whether it has an inverse using the "horizontal line test". If any horizontal line passes through only one point of the graph of *f*, then f^{-1} exists.

Proving two functions are inverses of each other

You can show whether or not two functions are inverses with the following test:

If both $f \circ g = x$ and $g \circ f = x$, then $g = f^{-1}$ and $f = g^{-1}$.

Note that you must do *both* compositions to show functions are inverses.

Finding the inverse of a function

The process for finding the inverse involves the idea behind the initial comments about inverses given above. Since

$x \longrightarrow f(x) \longrightarrow y$ and

$y \longrightarrow f^{-1}(y) \longrightarrow x$, we can derive the inverse (when it exists) and express it as a function of *x* by replacing *x* with *y* and *y* with *x* in the function definition and solving for the "new" *y*. For example, using $f(x) = 2x + 3$:

start: $y = 2x + 3$

replace: $x = 2y + 3$

solve for *y*: $-2y = -x + 3$

$$y = \frac{1}{2}x - \frac{3}{2}$$

$$f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$$

Note that we can check by using composition:

$$f \circ f^{-1} = f\left[\frac{1}{2}x - \frac{3}{2}\right] = 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3 = (x - 3) + 3 = x$$

$$f^{-1} \circ f = f^{-1}[2x + 3] = \frac{1}{2}(2x + 3) - \frac{3}{2} = \left(x + \frac{3}{2}\right) - \frac{3}{2} = x$$

Therefore, we do have the correct inverse function.