Supramaximal representations of planar surface groups

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Abstract

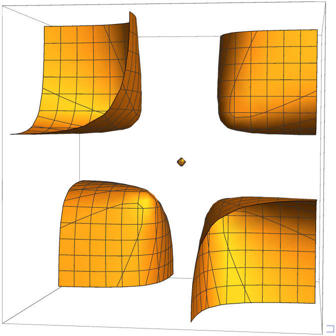
Recently Deroin, Tholozan and Toulisse found connected components of relative character varieties of surface group representations in a Hermitian Lie grop G with remarkable properties. For example, although the Lie groups are never compact, these components are compact. In this way they behave more like relative character varieties for compact Lie groups. (A relative character variety comprises equivalence classes of homomorphisms of the fundamental group of a surface S, where the holonomy around each boundary component of S is constrained to a fixed conjugacy class in G.) The first examples were found by Robert Benedetto and myself in an REU in summer 1992. Here S is the 4-holed sphere and G = SL(2,R). Although computer visualization played an important role in the discovery of these unexpected compact components, computation was invisible in the final proof, and its subsequent extensions.

A specific example

Fix $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^3$. Then the equation (in $(\xi, \eta, \zeta) \in \mathbb{R}^3$)

$$\xi^2 + \eta^2 + \zeta^2 + \xi\eta\zeta = (\alpha\beta + \gamma\delta)\xi \; + \; (\beta\gamma + \delta\alpha)\eta \; + \; (\alpha\gamma + \beta\delta)\zeta \; + 4 - (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \alpha\beta\gamma\delta),$$

describes a cubic surface in $\mathbb{R}^3$. For certain $(\alpha, \beta, \gamma, \delta)$ — for example, $(3/2, 3/2, 3/2, -3/2)$ — this surface has one component $\approx S^2$ and four components $\approx D^2$.



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  - Natural Poisson structure, relative character varieties are symplectic leaves, and the restriction maps are Casimirs.

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- The $\mathrm{Inn}\big(SL(2)\big)$-invariant mapping

$$\begin{aligned} \mathrm{Hom}(F_2, SL(2)) &\longrightarrow \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} \xi := & \mathrm{Tr}\big(\rho(X)\big) \\ \eta := & \mathrm{Tr}\big(\rho(Y)\big) \\ \zeta := & \mathrm{Tr}\big(\rho(XY)\big) \end{bmatrix} \end{aligned}$$

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- Characters in $\mathbb{R}^3 \subset \mathbb{C}^3$ (the $\mathbb{R}$-points) $\longleftrightarrow$ equivalence classes of representations into $\mathbb{R}$-forms $SU(2)$ and $SL(2, \mathbb{R})$ of $SL(2)$.

One-Holed Torus

The fundamental group of is a two-generator free group $\langle X, Y \rangle$ with redundant geometric presentation

$$\pi := \langle X, Y, K \mid K = XYX^{-1}Y^{-1} \rangle$$

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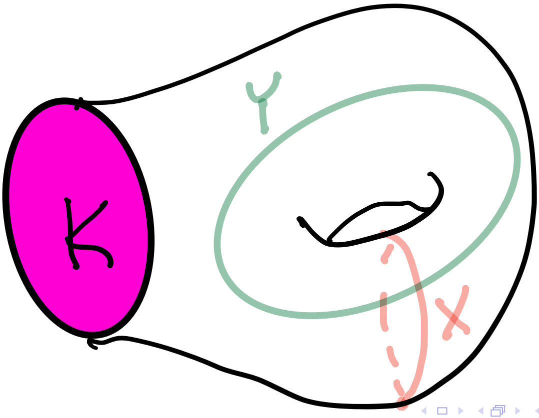

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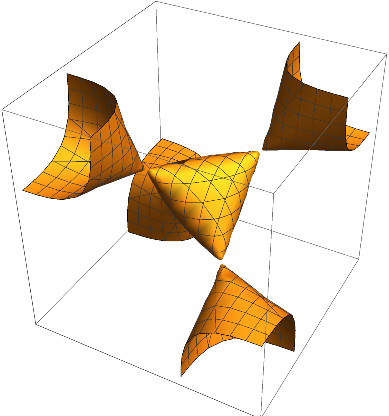
  - For $\kappa \geq 2$, level set connected noncompact.

Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$



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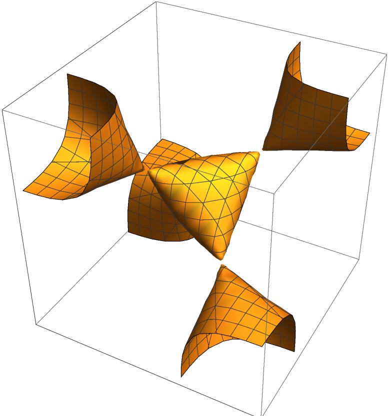

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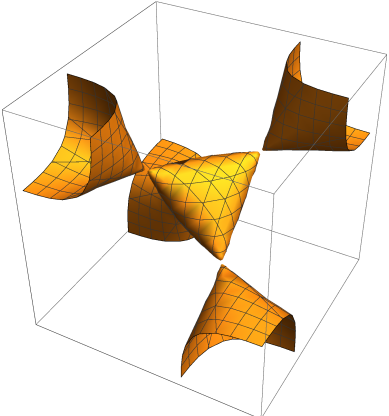
$$\xi = a + a^{-1}, \qquad \eta = b + b^{-1}, \qquad \zeta = ab + (ab)^{-1}$$



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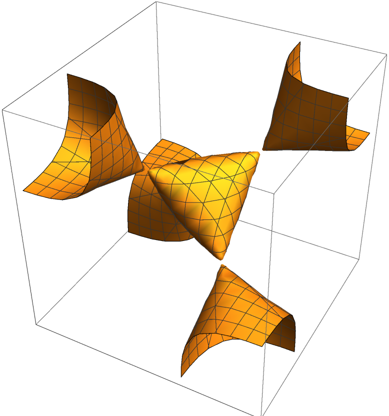
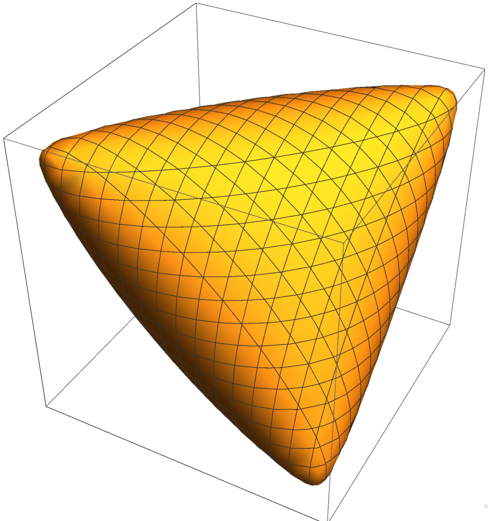
  - For example, $X \overset{\rho}{\longmapsto} \begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix}$, $Y \overset{\rho}{\longmapsto} \begin{bmatrix} b & * \\ 0 & b^{-1} \end{bmatrix}$.



$\mathbb{R}$-points: Unitary representations



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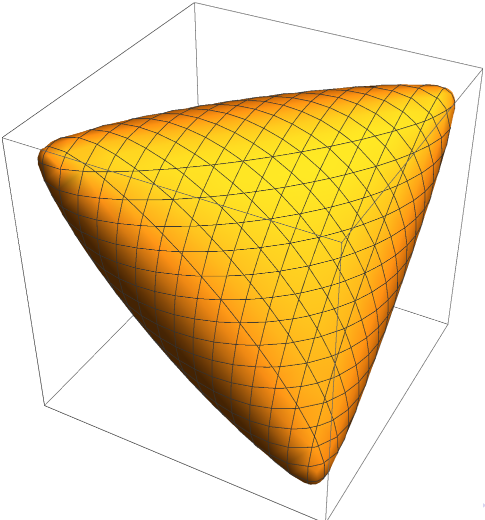
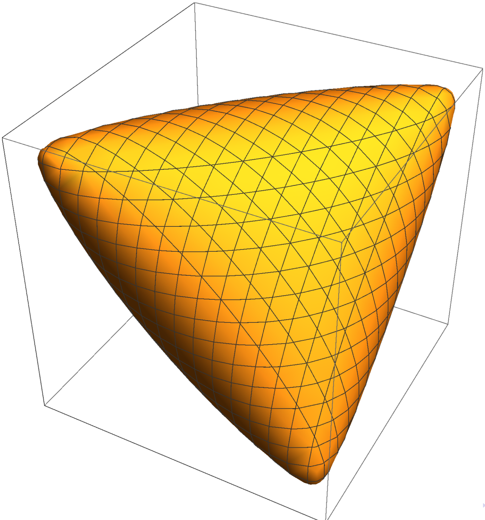
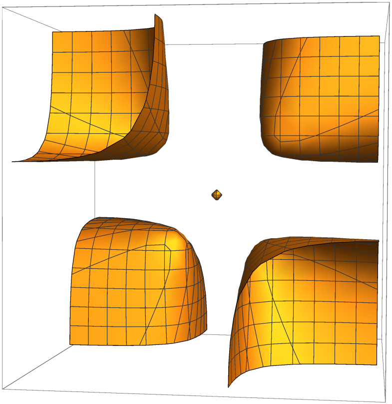


$\mathbb{R}$-points: Unitary representations

- $\mathbb{R}$-points correspond to representations into $\mathbb{R}$-forms of SL(2): either $\mathrm{SL}(2,\mathbb{R})$ or SU(2).
- Characters in $[-2,2]^3$ with $\kappa \leq 2 \longleftrightarrow$ SU(2)-representations.



The level set $\kappa = -2$: Markoff equation $\xi^2 + \eta^2 + \zeta^2 = \xi\eta\zeta$



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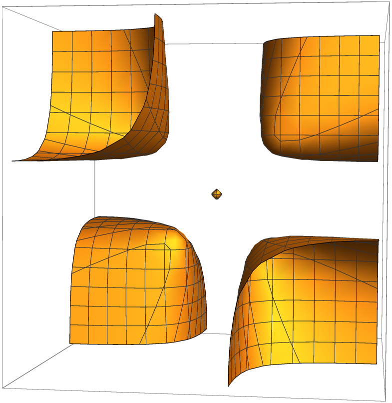

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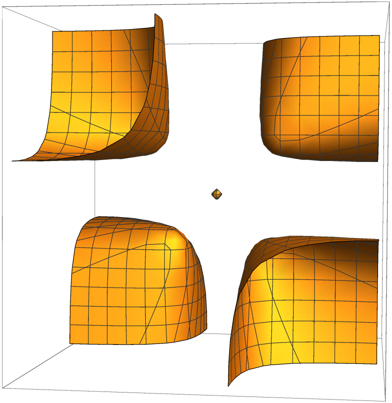

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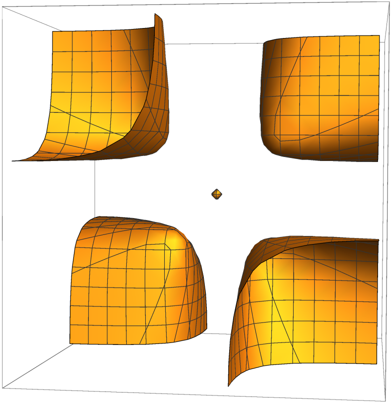
- Markoff surface $\mathbb{R}^3 \cap \kappa^{-1}(-2) \setminus \{O\}$ parametrizes complete finite area hyperbolic structures. on $\Sigma_{1,1}$, forming four other components.

Compact components for $\Sigma_{1,1}$



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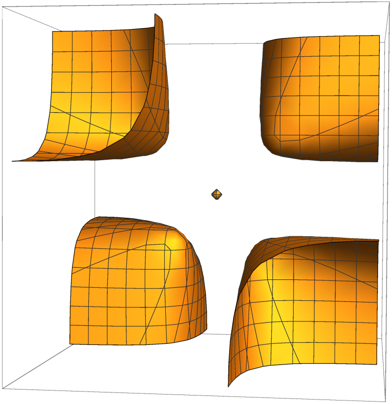

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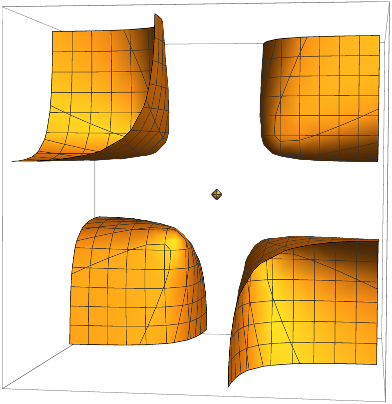

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- A fifth *compact* component corresponds to SU(2)-representations.

Four-Holed Sphere

The fundamental group of is a three-generator free group $\langle A, B, C\rangle$ given by redundant geometric presentation

$$\pi := \langle X, Y, Z, A, B, C, D \mid X = AB, Y = BC, Z = CA\rangle$$

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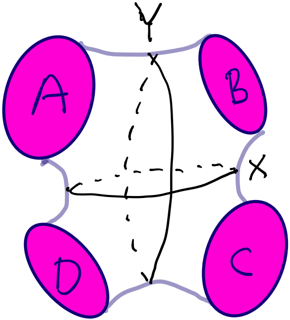

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  a family of cubic surfaces parametrized by $(\alpha, \beta, \gamma, \delta)$.
- (Benedetto-G 1992) For certain $(\alpha, \beta, \gamma, \delta) \in [-2, 2]$ *compact* components of $\mathrm{SL}(2, \mathbb{R})$-characters exist.

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- The orbit $\mathrm{Mod}(\Sigma)[\rho]$ is bounded.

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    - interpretation in terms of parabolic Higgs bundles (Biquard, Mondello), giving a holomorphic identification of the symplectic leaves as above.

References I

- R. Benedetto and W. Goldman, *The topology of the relative character variety of the quadruply-punctured sphere,* Experimental Mathematics (1999) **8:1**, 85–104.
- W. Goldman, "Trace coordinates on Fricke spaces of some simple hyperbolic surfaces," Chapter 15, pp. 611–684, of Handbook of Teichmüller theory, vol. II (A. Papadopoulos, ed.), IRMA Lectures in Mathematics and Theoretical Physics 13, EMS (2008), math.GM.0402103
- W. Goldman, Action of the modular group on real SL(2)-characters of a one-holed torus," Geometry and Topology 7 (2003), 443–486. mathDG/0305096.

References II

- M. Burger, A. Iozzi, and A. Wienhard, Surface group representations with maximal Toledo invariant, Ann. of Math. (2) 172 (2010), no. 1, 517–566.
- B. Deroin and N. Tholozan, Supra-maximal representations from fundamental groups of punctured spheres to $\mathrm{PSL}(2, \mathbb{R})$, Ann. Sci. Éc. Norm. Supér. (4) 52 (2019), no. 5, 1305–1329.
- G. Mondello, Topology of representation spaces of surface groups in $\mathrm{PSL}(2, \mathbb{R})$ with assigned boundary monodromy and nonzero Euler number. Pure Appl. Math. Q. 12 (2016), no. 3, 399–462.
- N. Tholozan and J. Toulisse, Compact connected components in relative character varieties of punctured spheres, Êpijournal de Géométrie Algébrique,`epigo.epsciences.org` Volue 5 (2021), Article No. 6, `GT.1811.01603v3`