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A REMARKABLE FAMIY OF AFFINE CUBIC SURFACES

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Abstract. In 1848 Cayley and Salmon proved that a nonsingular projective cubic over $\mathbb{C}$ contains 27 lines. The family of affine cubics defined by

$$x^2 + y^2 + z^2 - xyz = t + 2$$

arises in several contexts, including relative $\mathsf{SL}(2,\mathbb{C})$-character varieties of the one-holed torus $\mathbb{T}_1$ These relative character varieties enjoy a rich Poisson geometry, invariant under the mapping class group of $\mathbb{T}_1$. We describe their geometry, symmetry and dynamics, relating these to the structure of the classical geometry of their projective completion and complexification. We pay particular attention to the $\mathbb{R}$-levels when $t > 2$, when all the lines are real. However, the dynamics bifurcates at the level $t = 18$, where the level surface relates to the Clebsch diagonal surface.

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Date: September 4, 2024.

2000 *Mathematics Subject Classification.* 57M05 (Low-dimensional topology), 20H10 (Fuchsian groups and their generalizations).

Key words and phrases. character varieties, cubic surfaces.

Goldman gratefully acknowledges partial support from National Science Foundation grants DMS1709791 and the GEAR Research Network in the Mathematical Sciences DMS1107367, as well the following insitutions for their hospitality: Department of Mathematics at Brown University (Fall 2017), the Institute for Computational and Experimental Research in Mathematics (September 2019), the Mathematical Sciences Resarch Insitute (October–December 2019) and the Institute for Advanced Study (September 2021 - July 2022.

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Introduction

The trace $\mathsf{tr}[\xi,\eta]$ of the commutator $[\xi,\eta] = \xi\eta\xi^{-1}\eta^{-1}$ of two elements ξ,η of $\mathsf{SL}(2,\mathbb{C})$ defines a family of affine cubic surfaces

$$\mathcal{S}_t := \kappa^{-1}(t)$$

where

$$\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2 \tag{1}$$

and $x = \mathsf{tr}(\xi), y = \mathsf{tr}(\eta), z = \mathsf{tr}(\xi\eta)$. In homogeneous coordinates X, Y, Z, W where

$$(x, y, z) \longmapsto \left[\!\left[\begin{matrix} X \\ Y \\ Z \\ W \end{matrix}\right]\!\right] = \left[\!\left[\begin{matrix} x \\ y \\ z \\ 1 \end{matrix}\right]\!\right]$$

is the affine chart to the patch defined by $W \neq 0$,

$$x = X/W \tag{2}$$

$$y = Y/W \tag{3}$$

$$z = Z/W \tag{4}$$

are affine coordinates. Its projective completion $\overline{\mathbb{S}_t} \subset \mathsf{P}^3$ is defined by

$$(X^2 + Y^2 + Z^2)W - XYZ - (t+2)W^3 = 0 \tag{5}$$

When $t \neq \pm 2$, this surface is smooth.

$\overline{\mathbb{S}_{-2}}$ is the *Markoff cubic,* $\overline{\mathbb{S}_{-10/3}}$ is the *Fermat cubic,* $\overline{\mathbb{S}_2}$ is the *Cayley cubic,* and and $\overline{\mathbb{S}_{18}}$ is the *Clebsch cubic.* When $k \geq 2$, all the lines are real and we mainly concentrate on this case.

In 1848, Cayley proved that a smooth projective cubic surface $\overline{S}$ over $\mathbb{C}$ contains a line. He communicated this result to Salmon who shortly thereafter showed that $\overline{S}$ contains 27 lines. The purpose of this paper is to the geometry of the lines to study the dynamics of the modular group of $\mathbb{T}_1$ on the set of $\mathbb{R}$-points of the $\mathsf{SL}(2, \mathbb{C})$-character variety of $\mathbb{T}_1$.

Notations and terminology

We work over a field k of characterisitc zero, usually $\mathbb{R}$ or $\mathbb{C}$. We denote the affine space whose underlying vector space is k^n by A^n, or just A if n or $\mathbb{C}$ is understood from context. The vector space V can be reconstructed from A as the group of translations of A. If W is an vector space of dimension m, then the associated projective space $\mathsf{P}(\mathsf{W})$ is defined as the space of one-dimensional linear subspaces of W and has dimension $m - 1$. A basis of a one-dimensional subspace is just a nonzero vector, so points of $\mathsf{P}(\mathsf{W})$ can be defined as *projective equivalence classes* of nonzero vectors[1] in W.

[1]Two vectors are *projectively equivalent* if and only if they are nonzero scalar multiples of one another.

An affine space A^n embeds in a projective space $\mathsf{P}^n := \mathsf{P}(\mathsf{W})$, where $\mathsf{W} := V \oplus \mathsf{k}$, as the image of the affine hyperplane $\mathsf{V} \oplus \{1\}$ comprising vectors $\mathbf{v} \oplus 1$ in $\mathsf{V} \oplus \mathsf{k}$ under the quotient map

$$\mathsf{W} \setminus \{0\} \longrightarrow \mathsf{P}(\mathsf{W}).$$

For more details on affine and projective geometry, see Goldman [11].

We denote the symmetric group on n symbols by $\mathfrak{S}_n$.

if $S \subset \mathsf{A}^n$, denote its closure in P^n by $\overline{S}$.

A *line* on S (respectively $\overline{S}$) is an affine (respectively projective) line contained in S (respectively $\overline{S}$). The intersection of two lines on $\overline{S}$ will be called a *crossing point.* A crossing point is an *Eckardt point* (or an E-*point*) if it is the intersection of three distinct lines. A plane $P \subset \mathsf{A}^3$ is a *tritangent plane* (or simply a *tritangent*) if it is tangent to S at three points or an Eckardt point. It is called *generic* if its intersection with S is a union of three crossing lines; otherwise we call it an *Eckardt tritangent.* If T is a generic tritangent, then it contains three crossing points (the intersection of three lines) and equals the tangent plane T_pS for any of the crossing points p. If T is an Eckardt tritangent, then $T \cap S$ is the union of three distinct lines intersecting at an Eckardt point p and $T = \mathsf{T}_pS$.

In as yet unpublished work with Domingo Toledo [12], we show that given a projective cubic $\overline{S}$ together with a generic tritangent T, the affine piece $\overline{S} \setminus T$ can be defined as:

$$x^2 + y^2 + z^2 - xyz = px + qy + rz + s \tag{6}$$

and if it admits a certain symmetry (of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$) then the linear coefficients $p = q = r = 0$, and the affine cubic surface falls in our family $\mathcal{S}_t$, where$t = s + 2$. This condition is equivalent to the existence of 6 E-points, all ideal (that is, in T). In the general case, the cubic surface (6) corresponds to the relative character variety of a four-holed sphere.

The generic smooth projective cubic surface $\overline{S}$ over $\mathbb{C}$ contains 27 lines, 45 tritangents and 135 crossing points (and no E-points). A tritangent contains 27 crossing points, so the generic affine cubic surface S contains $108 = 135 - 27$ crossing points.

1. Singular points and symmetry

We begin with discussing the singular points of the level sets $\mathcal{S}_t = \kappa^{-1}(t)$. When $t \neq \pm 2$, $\mathcal{S}_t$ is smooth. The two singular level sets correspond to two famous cubic surfaces in our family: the level set $\mathcal{S}_2$ is the *Cayley cubic surface,* and $\mathcal{S}_t$ is the *Markoff cubic surface.* All the singular points are nodes: $\mathcal{S}_2$ has four nodes, which we call the *vertices* in this family and $\mathcal{S}_{-2}$ has a single node, the *origin.*

Then we discuss the group of automorphisms of κ. Since κ is a symmetric polynomial in the variables x, y, z, it admits $\mathfrak{S}_3$-symmetry. Furthermore changing the signs of two of the variables and keeping the third fixed defines *sign-change automorphisms,* forming a group we denote $\Delta \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Together $\mathfrak{S}_3$ and Δ generate a group $\mathfrak{S}_3 \rtimes \Delta$ which permutes the four vertices. Indeed every permutation of the four vertices is realized by an element of $\mathfrak{S}_3 \rtimes \Delta$. Therefore

$$\mathfrak{S}_3 \rtimes \Delta \cong \mathfrak{S}_4$$

and this group is the *linear automorphism group* of the surfaces $\mathcal{S}_t$. These are the automorphisms which extend to the projective completion $\overline{\mathcal{S}_t}$.

For listing the geometric objects, we exploit the 3-cycles in the cyclic *alternating group*

$$\mathfrak{A}_3 = \{(), (123), (132)\} < \mathfrak{S}_3.$$

1.1. **Critical points of κ.** Since

$$d\kappa = (2x - yz)dx + (2y - zx)dy + (2z - xy)dx,$$

the critical points of κ are the solutions to the system

$$2x - yz = 2y - zx = 2z - xy = 0.$$

If one of the variables is zero, then the other two vanish as well, obtaining the *origin*

$$\mathbf{o} := (0, 0, 0)$$

as a critical point with critical value $t = -2$.

Otherwise all variables are nonzero, and an elementary calculation shows that each quotient $x/y, y/z, z/x$ squares to 1, and their product equals 1. Thus each quotient equals ± 1 and either all of them equal 1 or exactly two of them equals -1. Applying a sign-change if necessary, we can assume $x = y = z$, in which case the common value equals 2. This gives four critical points of κ with critical value 2, which we call the *vertices:*

$$\begin{aligned}
\mathbf{c}_0 &:= (2, 2, 2);\\
\mathbf{c}_1 &:= (2, -2, -2);\\
\mathbf{c}_2 &:= (-2, 2, -2);\\
\mathbf{c}_3 &:= (-2, -2, 2).
\end{aligned} \tag{7}$$

These five critical points constitute the singular sets of the Markoff surface $\mathcal{S}_{-2}$ and the Cayley surface $\mathcal{S}_2$ respectively. They are all *nodes* (ordinary double points) of the respective level sets.

The origin $\mathbf{o} = (0,0,0)$ is the character of the *quaternion representation,* given by *Pauli matrices*

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

The vertices

$$\mathsf{Sing}(\mathcal{S}_{+2}) = \mathcal{C} := \{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$$

are characters of *central* representations, that is, representations $\mathsf{F}_2 \longrightarrow \{\pm\mathbb{I}_2\}$. These representations form a group

$$\Delta \;:=\; \mathsf{Hom}\big(\mathsf{F}_2, \{\pm\mathbb{I}_2\}\big) \;\cong\; \big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big),$$

acting simply transitively on $\mathcal{C}$.

These are exactly the linear *sign-change automorphisms* in $\mathsf{SL}(3,\mathbb{Z})$:

$$\delta_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \delta_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \delta_3 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In particular

$$\begin{aligned} &\mathbf{c}_0 \xleftrightarrow{\delta_1} \mathbf{c}_1, \qquad && \mathbf{c}_2 \xleftrightarrow{\delta_1} \mathbf{c}_3 \\ &\mathbf{c}_0 \xleftrightarrow{\delta_2} \mathbf{c}_2, && \mathbf{c}_3 \xleftrightarrow{\delta_2} \mathbf{c}_1 \\ &\mathbf{c}_0 \xleftrightarrow{\delta_3} \mathbf{c}_3, && \mathbf{c}_1 \xleftrightarrow{\delta_3} \mathbf{c}_2. \end{aligned}$$

and thus correspond to free involutions in $\mathfrak{S}_4 = \mathsf{Aut}(\mathcal{C})$ via:

$$\begin{aligned} \delta_1 &\longleftrightarrow (01)(23) \\ \delta_2 &\longleftrightarrow (02)(13) \\ \delta_3 &\longleftrightarrow (03)(12) \end{aligned}$$

2. The ideal locus

Now we discuss the closure $\overline{\mathcal{S}_t}$ of $\mathcal{S}_t \subset \mathsf{A}^3$ in P^3. The *ideal locus* $\overline{\mathcal{S}_t} \setminus \mathcal{S}_t$ consists of three crossing lines, which forms a tritangent plane to $\overline{\mathcal{S}_t}$. Its points are smooth points of $\overline{\mathcal{S}_t}$, although the intersection with $\mathsf{P}^2_\infty = \mathsf{P}^3 \setminus \mathsf{A}^3$ is not transverse.

Thus the 27 lines divide into 3 ideal lines and 24 finite lines.[2] The first step in our classification of lines uses these three lines to divide the finite lines into three 8-line families.

[2]These lines are counted with multiplicity in the singular cases $t = \pm 2$

2.1. **Locus at infinity.** To identify the ideal locus of $\mathcal{S}_t$, set $W = 0$ in the defining equation (5) for $\overline{\mathcal{S}_t}$, obtaining:

$$XYZ = 0.$$

This is the union of the three lines

$$\begin{aligned} {}^{X}\mathcal{I} &:= \{X = W = 0\} \\ {}^{Y}\mathcal{I} &:= \{Y = W = 0\} \\ {}^{Z}\mathcal{I} &:= \{Z = W = 0\}. \end{aligned}$$

These are the ideal loci of the three coordinate planes (or any parallel translates of them) in A^3.

These projective lines meet at the ideal points

$${}^{X}\mathbf{p}_\infty = \left[\!\!\left[\begin{matrix}1\\0\\0\\0\end{matrix}\right]\!\!\right], \qquad {}^{Y}\mathbf{p}_\infty = \left[\!\!\left[\begin{matrix}0\\1\\0\\0\end{matrix}\right]\!\!\right], \qquad {}^{Z}\mathbf{p}_\infty = \left[\!\!\left[\begin{matrix}0\\0\\1\\0\end{matrix}\right]\!\!\right]$$

At these crossing points, the ideal plane P_∞ (defined by $W = 0$) equals the tangent plane, so P_∞ is a generic tritangent to $\overline{\mathcal{S}_t}$, which we denote $\mathcal{T}_\infty$.

Our context is thus the family of projective cubics $\overline{\mathcal{S}_t}$ together with a fixed generic tritangent plane; see Goldman-Toledo [12]. These pairs $(\overline{\mathcal{S}_t}, \mathcal{T}_\infty)$ also exhibit extra symmetry, which fails in general. This symmetry is reflected in the absence of linear terms in the defining function κ, for $\mathcal{S}_t$.[3]

2.2. **Three families of non-ideal lines.** The basic fact we exploit is the following:

Theorem 2.2.1. *On each line $\ell \subset \mathcal{S}_t$, exactly one of the coordinate functions x, y, z is constant.*

Thus the finite lines fall into three families labeled by X, Y, or Z respectively. Each of the three families consists of of eight lines, grouped into four pairs.

We use the elementary fact:

[3]Alternatively, it is reflected in the existence of ideal Eckardt points (14). These linear terms do occur in the analogous description of the relative character variety of the 4-holed sphere, where the $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-symmetry is broken. See §3.1 for further details.

Lemma 2.2.2. *Suppose that* $\mathsf{A}^3 \xrightarrow{f} \mathsf{k}$ *is an affine function and* $\ell \subset \mathsf{A}^3$ *is a line. Let* F *be the covector in* $(\mathsf{k}^3)^*$ *extending* f *and let* λ *be the line in the ideal plane* P_∞ *corresponding to* $\mathsf{Ker}(F)$. *Then the ideal point of* ℓ *lies in* λ *if and only if the function* f *is constant on* ℓ.

Proof. Theorem 2.2.1 follows easily from Lemma 2.2.2 as follows. Each line ℓ on $\mathcal{S}_t$ has an ideal point

$$\ell_\infty := \bar{\ell} \cap \mathcal{T}_\infty.$$

Since the ideal locus

$$\overline{\mathcal{S}_t} \cap \mathcal{T}_\infty = {}^X\mathcal{I} \ \cup {}^Y\mathcal{I} \ \cup {}^Z\mathcal{I} \ ,$$

the ideal point ℓ_∞ must lie in an ideal coordinate line. Then $\ell_\infty \in {}^X\mathcal{I}$ if and only if the affine coordinate x is constant on ℓ (and similarly for y and z). ☐

A more pedestrian and intuitive way of seeing Theorem 2.2.1 is to imagine a line on $\mathcal{S}_t$ given in parametric form by:

$$s \longmapsto p(s) = (x_0 + s\xi, y_0 + s\eta, z_0 + s\zeta).$$

The composition $\kappa \circ p(s)$ is constant, yet it is given by a cubic polynomial in s with leading term $-s^3\xi\eta\zeta$. Letting $s \longrightarrow \infty$, we see that $\xi\eta\zeta = 0$, that is, one of the three coordinates x, y, z is constant on $p(s)$.

For example, the family corresponding to the Z-coordinate yields four planes in A^3 defined by

$$\begin{aligned} z_0 &= -\sqrt{t+2} \\ z_0 &= -2 \\ z_0 &= +2 \\ z_0 &= +\sqrt{t+2}. \end{aligned} \tag{8}$$

Each plane contains two lines, and the union with the ideal line Z_∞ is a tritangent. The lines in the planes $z_0 = \pm\sqrt{t+2}$ we label with "$\mathcal{C}$" (for *"crossing"*). The lines in the planes $z_0 \pm 2$ we label with "$\mathcal{P}$' for (for *"parallel"*). In general the intersection of $\mathcal{S}_k$ with plane $z = z_0$ is a conic, but this conic degenerates into a union of lines at the special levels (8). Thus each special level plane contains two lines, either parallel ($\mathcal{P}$) or crossing ($\mathcal{C}$).

This follows easily from writing the defining equation in terms of the family of quadratic forms:

$$\mathcal{Q}_z(x, y) := x^2 - zxy + y^2 \tag{9}$$

Then $\mathcal{S}_t$ is defined by:

$$\mathcal{Q}_z(x,y) = t + 2 - z^2 \tag{10}$$

because $\kappa(x,y,z) = \mathcal{Q}_z(x,y) - z^2 - 2$. When $z = \pm 2$,

$$\begin{aligned} \mathcal{Q}_2(x,y) &= (x-y)^2 \\ \mathcal{Q}_{-2}(x,y) &= (x+y)^2. \end{aligned} \tag{11}$$

When $z = \pm\sqrt{t+2}$,

$$\begin{aligned} \mathcal{Q}_{\sqrt{t+2}}(x,y) &= (y - m^+x)(y - m^-x) \\ \mathcal{Q}_{-\sqrt{t+2}}(x,y) &= (y + m^+x)(y + m^-x) \end{aligned} \tag{12}$$

where the two *slopes*, $m^\pm \in \mathbb{Q}[\sqrt{t+2}, \sqrt{t-2}]$ are defined by:

$$m^\pm := \frac{\sqrt{t+2} \pm \sqrt{t-2}}{2} \tag{13}$$

The slopes satisfy:

$$m^+m^- = 1, \qquad m^+ + m^- = \sqrt{t+2}, \qquad m^+ - m^- = \sqrt{t-2}.$$

3. Two types of non-ideal lines

We classify the 24 non-ideal lines into 12 $\mathcal{P}$-lines and 12 $\mathcal{C}$-lines. As in §2.2, these fall into three families, corresponding to the coordinates X, Y, Z. Each family contains 4 $\mathcal{P}$-lines and 4 $\mathcal{C}$-lines.

3.1. $\mathcal{P}$-lines.

The $\mathcal{P}$-lines arise when these lines are parallel, namely when $x_0 = \pm 2$, $y_0 = \pm 2$, or $z_0 = \pm 2$ respectively. The four $\mathcal{P}$-lines in the Z-family are:

$$\begin{array}{lll} {}^Z\mathcal{P}^+_+ : & z = +2 & y = x + \sqrt{t-2} \\ {}^Z\mathcal{P}^+_- : & z = +2 & y = x - \sqrt{t-2} \\ {}^Z\mathcal{P}^-_+ : & z = -2 & y = -x + \sqrt{t-2} \\ {}^Z\mathcal{P}^-_- : & z = -2 & y = -x - \sqrt{t-2} \end{array}$$

arising from the factorizations (11). Apply 3-cycles in $\mathfrak{A}_3$ to obtain similar formulas for the X-family and the Y-family of $\mathcal{P}$-lines.

3.1.1. *Ideal Eckard points.* These lines fall into six pairs of *parallelism classes,* namely ${}^X\mathcal{P}^{+}_{\pm}$, ${}^Y\mathcal{P}^{+}_{\pm}$, and ${}^Z\mathcal{P}^{+}_{\pm}$ respectively. They meet in six *ideal Eckardt points:*

$$
\begin{aligned}
&\mathsf{Eck}_\infty X+ \;:=\; {}^X\mathcal{P}^+_+ \cap {}^X\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}0\\1\\1\\0\end{matrix}\right]\!\!\right]. \qquad
&&\mathsf{Eck}_\infty X- \;:=\; {}^X\mathcal{P}^-_+ \cap {}^X\mathcal{P}^-_-. = \left[\!\!\left[\begin{matrix}0\\1\\-1\\0\end{matrix}\right]\!\!\right]. \\
&\mathsf{Eck}_\infty Y+ \;:=\; {}^Y\mathcal{P}^+_+ \cap {}^Y\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}1\\0\\1\\0\end{matrix}\right]\!\!\right]. \qquad
&&\mathsf{Eck}_\infty Y- \;:=\; {}^Y\mathcal{P}^-_+ \cap {}^Y\mathcal{P}^-_- = \left[\!\!\left[\begin{matrix}1\\0\\-1\\0\end{matrix}\right]\!\!\right]. \\
&\mathsf{Eck}_\infty Z+ \;:=\; {}^Z\mathcal{P}^+_+ \cap {}^Z\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}1\\1\\0\\0\end{matrix}\right]\!\!\right]. \qquad
&&\mathsf{Eck}_\infty Z- \;:=\; {}^Z\mathcal{P}^-_+ \cap {}^Z\mathcal{P}^-_- = \left[\!\!\left[\begin{matrix}1\\-1\\0\\0\end{matrix}\right]\!\!\right].
\end{aligned}
\tag{14}
$$

3.2. **$\mathcal{C}$-lines.** $\mathcal{C}$-lines arise when the plane section is a degenerate conic, consisting of a a pair of *crossing* lines. Consider the $\mathcal{C}$-lines in the Z-family. Rewrite the defining equation for $\mathcal{S}_t$ as (10) using the family of quadratic forms $\mathcal{Q}_z$, defined in (9). Thus the $\mathcal{C}$-lines occur at the levels $z = z_0$ where $\mathcal{Q}_{z_0}(x, y) = 0$, that is, when $z^2 = t + 2$. On these levels

$$z_0 = +\sqrt{t+2},$$

the factorization (12) implies that the plane section is the union of lines

$$y = m^{\pm} x$$

where the slopes $m^{\pm}$ are defined in (13). Similarly on the level $z_0 = -\sqrt{t+2}$, the conic defined by $\mathcal{Q}_{z_0}$ degenerates into the pair of lines

$$y = -m^{\pm} x$$

Thus the four $\mathcal{C}$-lines in the Z-family are:

$$\begin{aligned}
{}^{Z}\mathcal{C}^{+}_{+}: &\qquad z = +\sqrt{t+2} &\qquad y = m^{+}x \\
{}^{Z}\mathcal{C}^{+}_{-}: &\qquad z = +\sqrt{t+2} &\qquad y = m^{-}x \\
{}^{Z}\mathcal{C}^{-}_{+}: &\qquad z = -\sqrt{t+2} &\qquad y = -m^{+}x \\
{}^{Z}\mathcal{C}^{-}_{-}: &\qquad z = -\sqrt{t+2} &\qquad y = -m^{-}x
\end{aligned}$$

3.2.1. *Dihedral characters.* As with $\mathcal{P}$-lines, the $\mathcal{C}$-lines naturally pair; however the $\mathcal{C}$-lines *cross* rather than being parallel. These crossing points play an important role in the geometry, and also arise as the fixed points of the sign-change group Δ. They arise as the characters of irreducible $\mathsf{SL}(2,\mathbb{C})$-representations which are not *absolutely irreducible,* since their restrictions to index-two subgroups are reducible. Since their images are dihedral groups, we call such points *dihedral characters* or D-points.

For fixed t, the six D-points are:

$$\begin{aligned}
{}^{X}\mathsf{Dih}^{\pm} &= (\pm\sqrt{t+2}, 0, 0) \\
{}^{Y}\mathsf{Dih}^{\pm} &= (0, \pm\sqrt{t+2}, 0) \\
{}^{Z}\mathsf{Dih}^{\pm} &= (0, 0, \pm\sqrt{t+2})
\end{aligned}$$

These are the fixed points on $\mathcal{S}_t$ of the sign-changes $\delta_1, \delta_2, \delta_3$ respectively.

The corresponding representation of $\mathbb{F}_2$ has ξ, η symmetries in points and their product $\xi\eta$ a transvection along the geodesic joining $\mathsf{Fix}(\xi)$ and $\mathsf{Fix}(\eta)$ in H^3. For example, ${}^{Z}\mathsf{Dih}^{+} = (0, 0, \sqrt{t+2})$ corresponds to:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{t+2} \\ -1/\sqrt{t+2} & 0 \end{bmatrix}, \begin{bmatrix} m^{-} & 0 \\ 0 & m^{+} \end{bmatrix}$$

where the slopes $m^{\pm}$ are defined in (13).

Since D-points are crossing points, their tangent planes are tritangents. Their intersections with $\overline{\mathcal{S}_t}$ are the two crossing $\mathcal{C}$-lines togther with the ideal line corresponding to their coordinate family.

In the example above, ${}^{Z}\mathcal{C}^{+}_{\pm}$, together with ${}^{Z}\mathcal{I}$, span the tritangent plane extending $z = +\sqrt{t+2}$, which meets $\overline{\mathcal{S}_t}$ in the union

$${}^{Z}\mathcal{C}^{+}_{+} \cup {}^{Z}\mathcal{C}^{+}_{-} \cup {}^{Z}\mathcal{I} \ .$$

The dihedral characters form a 6-element subset of $\mathcal{S}_t$, invariant under its full automorphism group. When $t \neq -2$, they crossing points, which all coalesce to the origin $\mathbf{o}$ as $t \longrightarrow -2$.

4. Sylvester's pentahedron

We give an alternate form for the cubic surfaces $\overline{\mathcal{S}_t} \subset \mathsf{P}^3$ using a projective embedding $\mathsf{P}^3 \hookrightarrow \mathsf{P}^4$ which will clarify extra symmetry of $\mathfrak{S}_4$ for the level set for $t = 18$,

identifying it as the Clebsch surface. Along the way, we find pentahedral forms for the Fermat surface, the Markoff surface and the Cayley surface

4.1. **Pentahedral form.** Let $\mathsf{P}^3_1 \subset \mathsf{P}^4$ be the projective hyperplane defined in homogeneous coordinates U_0, U_1, U_2, U_3, U_4 by

$$U_0 + U_1 + U_2 + U_3 + U_4 = 0. \tag{15}$$

To describe the embeddings $\overline{\mathcal{S}_t} \hookrightarrow \mathsf{P}^3_1$ we introduce a linear embedding $\mathbb{C}^3 \xrightarrow{\mathbf{U}} \mathbb{C}^4$ whose image is the hyperplane defined by (15):

$$\begin{aligned} U_0 &:= W \\ U_1 &:= (-2W + X - Y - Z)/8 \\ U_2 &:= (-2W - X + Y - Z)/8 \\ U_3 &:= (-2W - X - Y + Z)/8 \\ U_4 &:= (-2W + X + Y + Z)/8 \end{aligned} \tag{16}$$

Clearly the covectors in $\mathbb{C}^4$ satisfy (15), and thus $\mathbf{U}$ defines a projective embedding $\mathsf{P}^3 \hookrightarrow \mathsf{P}^4$ whose image is P^3_1. With more work, one calculates:

$$\begin{aligned} &(X^2 + Y^2 + Z^2)W - XYZ - (t+2)W^3 = \\ &\qquad \frac{-64}{3}\left\{\left(\frac{3t+10}{64}\right)(U_0)^3 + (U_1)^3 I + (U_2)^3 + (U_3)^3 + (U_4)^3\right\} \end{aligned}$$

Thus $\mathbf{U}$ maps $\overline{\mathcal{S}_t}$ to the subvariety in P^3_1 defined in homogeneous coordinates by: U_0, U_1, U_2, U_3, U_4 on P^3_1 by

$$\left(\frac{3t+10}{64}\right)(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \tag{17}$$

This is *Sylvester's pentahedral form* for $\overline{\mathcal{S}_t}$.

The five covectors U_0, U_1, U_2, U_3, U_4 define the *Sylvester planes:*

$$\mathsf{Syl}_i := \{U_i = 0\}$$

for $i = 0, 1, 2, 3, 4$. The plane

$$\mathsf{Syl}_0 = \{W = 0\}$$

equals the ideal tritangent $\mathcal{T}_\infty$, but in general Syl_i is *not* tritangent unless $t = 18$. The *ideal complete quadrilateral* is the configuration in $\mathcal{T}_\infty \approx \mathsf{P}^2$ defined by the four lines

$$f_i := \mathsf{Syl}_0 \cap \mathsf{Syl}_i$$

for $i = 1, 2, 3, 4$.

4.2. **Some special cases.** For the respective cases $t = 18, 2, -2, -10/3$, the coefficient $(3t+10)/64$ of U_0 assumes the respective values $1, 1/4, 1/16, 0$. These correspond to the Clebsch surface, the Cayley surface, the Markoff surface, and the Fermat surface, respectively.

4.2.1. *The Markoff surface.*

4.2.2. *The Cayley surface.* When $t = 2$, the surface admits a rational parametrization, as the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution $(a, b) \longmapsto (a^{-1}, b^{-1})$. Specifically, the branched double covering

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \mathcal{S}_2 \\ (a, b) &\longmapsto (a + a^{-1}, b + b^{-1}, ab + a^{-1}b^{-1}) \end{aligned}$$

is a quotient mapping. These correspond to the characters of *reducible* $\mathsf{SL}(2, \mathbb{C})$-representations, where X, Y, Z map to the diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}, \begin{bmatrix} a^{-1}b^{-1} & 0 \\ 0 & ab \end{bmatrix}$$

respectively. The *central* representations (where the image lies in the center $\pm\mathbf{1}$ of $\mathsf{SL}(2, \mathbb{C})$) correspond to its four nodes $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ (or *vertices*) of $\mathcal{S}_2$. (Compare [10].) These are four of the five critical points of κ.

4.2.3. *The Fermat surface.* This is the cubic surface in $\mathbb{C}\mathsf{P}^3$ defined in homogeneous coordinates by:

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 = 0. \tag{18}$$

As this defining polynomial in 4 homogeneous coordinates is symmetric, it admits the symmetry of $\mathfrak{S}_4$. Scalar multiplications of the homogeneous coordinates by cube roots of unity yield an action of $(\mathbb{Z}/3)^3$ by diagonal 4×4 matrices, and these actions generate the full automorphism group $\mathsf{Aut}(S) = (\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$ which has order

$$\#\mathsf{Aut}(S) = 3^3 \cdot 4! = 648.$$

Since the cube roots of unity are not real, the $\mathbb{R}$-locus only admits the $\mathfrak{S}_4$-symmetry (24 automorphisms).

To find a generic tritangent, combine (18) with the identity

$$U_1^3 + U_2^3 + U_3^3 + (-U_1 - U_2 - U_3)^3 = -3(U_1 + U_2)(U_2 + U_3)(U_3 + U_1)$$

to see that the plane P defined by $U_0 + U_1 + U_2 + U_3 = 0$ meets the surface in the union of three lines defined by:

$$(U_1 + U_2)(U_2 + U_3)(U_3 + U_1) = 0$$

and is a generic tritangent.

4.2.4. *The Clebsch surface.* When $t = 18$, the cubic polynomial

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3$$

is symmetric and the level set $\overline{\mathcal{S}_t}$ admits an obvious symmetry of the 120-element symmetric group $\mathfrak{S}_5$. This surface has four *finite Eckard points* and will be discussed extensively in §9.

5. Symmetry

We first describe the automorphisms of these cubics. The *linear automorphisms* form the symmetric group $\mathfrak{S}_4$, which extend to projective automorphisms of $\overline{\mathcal{S}_t}$. We first describe this action, which combines the triple symmetry of the three variables with the sign-change group described above. The group $\mathfrak{S}_4$ arises concretely as automorphisms of several natural 4-element sets:

- The *vertices,* that is, the *nodes* of the Cayley cubic $\mathcal{S}_2$;
- The ends of the real locus $\mathbb{R}^3 \cap \mathcal{S}_t$;
- Complementary ideal triangular regions $\overline{\mathcal{S}_t}$, that is, the four components of the complement of the ideal locus

  $$^X\mathcal{I} \cup {}^Y\mathcal{I} \cup {}^Z\mathcal{I} \subset \mathcal{T}_\infty.$$

  in the ideal tritangent plane.
- The finite E-points $\{\mathsf{Eck}_0, \mathsf{Eck}_1, \mathsf{Eck}_2, \mathsf{Eck}_3\}$ of $\mathcal{S}_{18}$. These will be discussed in detail in §9.

5.1. Linear automorphisms.

These cubics all exhibit a finite symmetry group $\mathfrak{S}_4$, which can be realized as the group of *all* automorphisms of the four-element set $\mathcal{C}$, which are the vertices of a natural *tetrahedron* on the Cayley cubic $\mathcal{S}_{+2}$.

Observe first that these cubics are symmetric in the three coordinates X, Y, Z, leading to an action of the symmetric group $\mathfrak{S}_3$, which we will heavily exploit. The permutations of coordinates lead to permutations of $\mathcal{C}$ which fix $\mathbf{c}_0$ but permute $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. The rest of $\mathfrak{S}_4$ can be understood in terms of the group Δ of *sign-change automorphisms,* described above.

In terms of coordinates, the symmetric group is a split extension

$$\Delta \ \triangleleft \ \mathfrak{S}_4 \ \twoheadrightarrow \ \mathfrak{S}_3,$$

where the epimorphism $\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$ is realized by permuting the three coordinates, and the kernel Δ comprises sign-changes as above. For representations, the sign-changes correspond to the action of the group of central representations $\mathsf{Hom}\big(\mathsf{F}_2, \{\pm\mathbf{1}\}\big)$ on $\mathsf{Hom}\big(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})\big)$. The orbits comprise lifts of $\mathsf{PSL}(2,\mathbb{C})$-representations to $\mathsf{SL}(2,\mathbb{C})$ and the relative character variety $\mathcal{S}_t$ corresponds to the image of a $\Delta\cdot\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-invariant subset of $\mathsf{Hom}\big(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})\big)$ under the $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-quotient map.

The symmetric group $\mathfrak{S}_4$ is the group of *automorphisms* of the projective cubic $\overline{\mathcal{S}_t}$ for generic t, and is realized as above by *linear automorphisms* of $\mathcal{S}_t$. The ends of the real level sets $\overline{\mathcal{S}_t} \cap \mathbb{R}^3$ form a four-element set whose full automorphism group equals $\mathfrak{S}_4$.

Another finite subset invariant under the automorphism group $\mathfrak{S}_4$ is the six-element subset of *ideal Eckardt points* defined in (14) of §3.1. These are the ideal points of the $\mathcal{P}$-lines.

As for the six-element subset comprising dihedral characters, $\mathfrak{S}_4$ is the centralizer of an involution in $\mathfrak{S}_6$ corresponding to the six-element subset consisting of ordered pairs of distinct points of $\{0,1,2,3\}$.

5.2. **Vieta invoutions.** In addtion to the finite groups of automorphisms which extend to projective automorphisms, the affine cubics $\mathcal{S}_t$ admit infinite groups of symmetries defining interesting dynamical systems.

Namely, the coordinate projections $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ define double (branched coverings) of $\mathcal{S}_t$, and their Galois groups generate an action of the *free* 3*-generator Coxeter group* $\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2$. That these automorphisms generate the full group of automorphisms of the affine cubic surfaces $\mathcal{S}_t$ is due to [5].

Take, for example, the coordinate projection for the z-coordinate. Fix $x_0, y_0 \in \mathbb{C}$. Then the restriction of the defining cubic polynomial κ to the coordinate line $\{(x_0,y_0)\} \times \mathbb{C}$ is quadratic. Thus the intersection

$$\mathcal{S}_t \ \cap \quad \{(x_0,y_0)\} \times \mathbb{C}$$

corresponds to the pair of solutions z of the quadratic equation

$$t \ = \ \kappa(x_0,y_0,z) \ = \ z^2 \ - \ (x_0 y_0)\, z \ + \ (x^2 + y^2 - 2).$$

If z, z' are the two solutions, then $z + z' = x_0 y_0$, so

$$z' = x_0 y_0 - z.$$

The deck transformation of the double covering $\mathcal{S}_t \longrightarrow \mathbb{C}^2$ is the *Vieta involution.* The three Vieta involutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^X\nu} \begin{bmatrix} x' := yz - x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^Y\nu} \begin{bmatrix} x \\ y' := zx - y \\ z \end{bmatrix},$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^Z\nu} \begin{bmatrix} x \\ y \\ z' := xy - z \end{bmatrix}.$$

They generate a free Coxeter group $\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2.$, naturally isomorphic to the level 2 congruence subgroup $\mathsf{PGL}(2,\mathbb{Z})_{(2)}$. Specifically, the respective Vieta involutions ${}^Z\nu$, ${}^X\nu$, ${}^Y\nu$ are realized by the automorphisms of $\mathbb{F}_2$ and the corresponding elements of $\mathsf{PGL}(2,\mathbb{Z})$ acting on points z in the upper half-plane, respectively:

$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \xmapsto{{}^Z\nu} \begin{pmatrix} X \\ Y^{-1} \\ YX^{-1} \end{pmatrix}\right] \longleftrightarrow \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : \left(z \longmapsto -\bar{z}\right)$$
$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \xmapsto{{}^X\nu} \begin{pmatrix} Y^{-1}X^{-1}Y^{-1} \\ Y \\ Z^{-1} \end{pmatrix}\right] \longleftrightarrow \pm \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} : \left(z \longmapsto \frac{\bar{z}}{1 - 2\bar{z}}\right)$$
$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \xmapsto{{}^Y\nu} \begin{pmatrix} X^{-1} \\ X^2Y^{-1} \\ Z \end{pmatrix}\right] \longleftrightarrow \pm \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} : \left(z \longmapsto 2 - \bar{z}\right).$$

The realizations in $\mathsf{PGL}(2,\mathbb{Z})$ are reflections in the imaginary axis $i\mathbb{R}_+$, the unit semicircle $|z| = 1$, and the vertical line $1 + i\mathbb{R}_+$, respectively. (Compare [13].)

6. Automorphisms and lines

We now describe the action of $\mathfrak{S}_4$ on the set of lines.

A sign-change leaves invariant each line in the corresponding family of $\mathcal{C}$-lines. Then, for example the lines in the Z-family

$$
\begin{aligned}
{}^{Z}\mathcal{C}^{+}_{\pm} &= \{z = +\sqrt{t+2},\ y = m^{\pm}x\} \\
{}^{Z}\mathcal{C}^{-}_{\pm} &= \{z = -\sqrt{t+2},\ y = -m^{\pm}x\}
\end{aligned}
$$

are each invariant under $(x,y,z) \mapsto (-x,-y,z)$ which corresponds to the transposition $\sigma^{(03)} \in \mathfrak{S}_4$. The transposition $\sigma^{(12)}$ interchanges

$$ {}^{Z}\mathcal{C}^{\pm}_{+} \leftrightarrow {}^{Z}\mathcal{C}^{\pm}_{-}. $$

The two remaining sign-changes (corresponding to $\sigma^{(01)(23)}$ and $\sigma^{(02)(13)}$ in $\mathfrak{S}_4$) interchange

$$ {}^{Z}\mathcal{C}^{-}_{\pm} \leftrightarrow {}^{Z}\mathcal{C}^{+}_{\pm}. $$

Here is what happens for $\mathcal{P}$-lines. Consider the Z-family. Each

$$ {}^{Z}\mathcal{P}^{-}_{\pm} = \{z = -2,\ x + y = \pm\sqrt{t-2}\} $$

is invariant under the transposition (12). The sign-change δ_3 interchanges these two lines; the other sign-changes interchange

$$ {}^{Z}\mathcal{P}^{-}_{\pm} \longleftrightarrow {}^{Z}\mathcal{P}^{+}_{\pm}. $$

For ${}^{Z}\mathcal{P}^{+}_{\pm} = \{z = +2,\ y - x = \pm\sqrt{t-2}\}$, the transposition

$$
\begin{aligned}
(03) &= \delta_3(12) = (12)\delta_3 \\
&= \delta_1(12)\delta_1 = \delta_2(12)\delta_2 \in \mathfrak{S}_4
\end{aligned}
$$

is realized by $(x,y,z) \mapsto (-y,-x,z)$ and leaves each of these two lines invariant.

6.1. **Involutions and lines.** Indeed, the involutions in $\mathfrak{S}_4$ distinguish the $\mathcal{P}$-lines from the $\mathcal{C}$-lines as follows. The *alternating group* $\mathfrak{A}_4 < \mathfrak{S}_4$ consists of even permutations. The nontrivial elements of the sign-change group $\Sigma < \mathfrak{A}_4$ consists of even permutations of order two, namely products of disjoint transpositions

$$ \sigma^{(ij)(l4)} \leftrightarrow \delta_l $$

where $\{i,j,l\} = \{1,2,3\}$. The odd permutations of order two are the transpositions $\sigma^{(ij)}$ where $\{i,j\}$ is a 2-element subset of $\{1,2,3\}$ and

$$ \sigma^{(l4)} \leftrightarrow \delta_l \circ \sigma^{(ij)}. $$

Proposition 6.1.1. *Let $\ell \subset \mathcal{S}_t$ be a (non-ideal) line. Then ℓ is invariant under an odd involution $\Longleftrightarrow$ ℓ is a $\mathcal{P}$-line, and ℓ is invariant under an even involution $\Longleftrightarrow$ ℓ is a $\mathcal{C}$-line.*

The action of $\mathfrak{S}_4$ on the ideal lines in $\overline{\mathbb{S}_t}$ is even easier. Each sign-change in Δ fixes each ideal line pointwise. The subgroup

$$\mathfrak{S}_3 = \mathsf{Aut}(\{1,2,3\}) < \mathfrak{S}_4$$

complementary to $\Delta \triangleleft \mathfrak{S}_4$ acts by permutations of $\{{}^X\mathcal{I}\ , {}^Y\mathcal{I}\ , {}^Z\mathcal{I}\ \}$.

6.2. **Vieta automorphisms and lines.** The Vieta involutions do not extend to projective space (they are not even defined by homogeneous polynomials), and therefore do not act on the ideal lines. We describe their action on a sample $\mathcal{P}$-line and a sample $\mathcal{C}$-line. As $\mathfrak{S}_4$ acts transitively on the set of $\mathcal{P}$-lines (respectively $\mathcal{C}$-lines), it suffices for the discussion to consider one sample line from each type.

First consider the $\mathcal{P}$-line ${}^Z\mathcal{P}^+_+$. The action of ${}^X\nu$ on it is:

$$\begin{bmatrix} x \\ x-\sqrt{t-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x-2\sqrt{t-2} \\ \big(x-2\sqrt{t-2}\big)+2\sqrt{t-2} \\ 2 \end{bmatrix}$$

and the ${}^Y\nu$ -action is:

$$\begin{bmatrix} x \\ x-\sqrt{t-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x \\ x+\sqrt{t-2} \\ 2 \end{bmatrix}$$

Since both involutions map ${}^Z\mathcal{P}^+_\pm \to {}^Z\mathcal{P}^+_\mp$, their composition ${}^X\nu \circ {}^Y\nu$ preserves each ${}^Z\mathcal{P}^+_\pm$, translating by $-\sqrt{t-2}$ on each.

The involution ${}^Z\nu$ maps ${}^Z\mathcal{P}^+_+ +$ to a parabola:

$$\begin{bmatrix} x \\ x-\sqrt{t-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x \\ x-\sqrt{t-2} \\ \big(x+\frac{\sqrt{t-2}}{2}\big)^2-\frac{k+6}{4} \end{bmatrix}$$

Next consider the $\mathcal{C}$-line ${}^Z\mathcal{C}^+_+$. Both ${}^X\nu$ and ${}^Y\nu$ map ${}^Z\mathcal{C}^+_+ \longrightarrow {}^Z\mathcal{C}^+_-$:

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{t+2} \end{bmatrix} \xmapsto{{}^X\nu} \begin{bmatrix} (m^+)^2x \\ m^+x \\ +2\sqrt{t+2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{t+2} \end{bmatrix} \overset{{}^Y\nu}{\longmapsto} \begin{bmatrix} x \\ m^-x \\ +\sqrt{t+2} \end{bmatrix}$$

Since both involutions map ${}^Z\mathcal{C}^+_+ \to {}^Z\mathcal{C}^+_-$, their composition ${}^X\nu \circ {}^Y\nu$ preserves ${}^Z\mathcal{C}^+_+$, scaling by $(m^-)^2$.

The involution ${}^Z\nu$ maps ${}^Z\mathcal{C}^+_++$ to a parabola:

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{t+2} \end{bmatrix} \longmapsto \begin{bmatrix} x \\ m^+x \\ m^+x^2 - \sqrt{t+2} \end{bmatrix}$$

6.3. **Degeneration of lines: the Markoff surface $t = -2$.** When $t \to -2$, the levels at $\pm\sqrt{t+2}$ coalesce at the level 0, and each pair ${}^\circ\mathcal{C}^+_\pm, {}^\circ\mathcal{C}^-_\pm$ converge to a single line, for example:

$$\begin{aligned} {}^X\mathcal{C}_\pm &:= \{z = \pm iy, x = 0.\} \\ {}^Y\mathcal{C}_\pm &:= \{x = \pm iz, y = 0.\} \\ {}^Z\mathcal{C}_\pm &:= \{y = \pm ix, z = 0.\} \end{aligned}$$

This gives 6 $\mathcal{C}$*-lines,* each counted with multiplicity 2. The remaining 12 $\mathcal{P}$-lines are:

$$\begin{aligned} {}^X\mathcal{P}^+_\pm &= \{z = y \pm 2i, x = +2\} \\ {}^X\mathcal{P}^-_\pm &= \{z = -y \pm 2i, x = -2\} \\ {}^Y\mathcal{P}^+_\pm &= \{x = z \pm 2i, y = +2\} \\ {}^Y\mathcal{P}^-_\pm &= \{x = -z \pm 2i, y = -2\} \\ {}^Z\mathcal{P}^+_\pm &= \{y = x \pm 2i, z = +2\} \\ {}^Z\mathcal{P}^-_\pm &= \{y = -x \pm 2i, z = -2\} \end{aligned}$$

This gives 6 double $\mathcal{C}$-lines, 12 $\mathcal{P}$-lines and 3 ideal lines, verifying the total count of 27 lines with multiplicity. The singularity at **o** is the concurrent intersection of three double lines.

6.4. **Degeneration of lines: the Cayley surface $t = +2$.** The degeneration is more severe on $\mathcal{S}_{+2}$. In that case, $\sqrt{t-2} = 0$ implies that all the $\mathcal{P}$-lines $\mathcal{P}^+_\pm$ (respectively $\mathcal{P}^-_\pm$) coalesce. Furthermore since $m^+ = m^- = 1$, the $\mathcal{C}$-lines $\mathcal{C}^+_\pm$ (respectively $\mathcal{C}^-_\pm$) coalesce. There remain

6 *quadruple lines:*

$$
\begin{aligned}
{}^X\mathcal{P}^+ = {}^X\mathcal{C}^+ &= \{z = y, x = +2\}\\
{}^X\mathcal{P}^- = {}^X\mathcal{C}^- &= \{z = -y, x = -2\}\\
{}^Y\mathcal{P}^+ = {}^Y\mathcal{C}^+ &= \{x = z, y = +2\}\\
{}^Y\mathcal{P}^- = {}^Y\mathcal{C}^- &= \{x = -z, y = -2\}\\
{}^Z\mathcal{P}^+ = {}^Z\mathcal{C}^+ &= \{y = x, z = +2\}\\
{}^Z\mathcal{P}^- = {}^Z\mathcal{C}^- &= \{y = -x, z = -2\}
\end{aligned}
$$

7. Galois automorphisms

The lines, tritangent planes, and their intersections also enjoy *Galois symmetry* as follows. Their coordinates lie in the biquadratic field $\mathbb{Q}[\sqrt{t+2}, \sqrt{t-2}]$, at least when $\sqrt{t \pm 2} \notin \mathbb{Q}$. Its Galois group is generated by involutions

$$
\sqrt{t+2} \overset{\mathcal{G}^+}{\longleftrightarrow} -\sqrt{t+2}, \qquad \sqrt{t-2} \overset{\mathcal{G}^-}{\longleftrightarrow} -\sqrt{t-2}.
$$

This group, also isomorphic to $\big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big)$, acts on the configuration of lines, tritangent planes, and intersection points.

Each pair of lines in one of the three coordinate families (X, Y or Z) and four levels (8) is interchanged by the Galois involution $\mathcal{G}^-$ (see below). Observe that

$$
m^\pm \overset{\mathcal{G}^-}{\longleftrightarrow} m^\mp, \qquad m^\pm \overset{\mathcal{G}^+}{\longleftrightarrow} -m^\mp
$$

and $\mathcal{G}^+ \circ \mathcal{G}^- = \mathcal{G}^- \circ \mathcal{G}^+$ is an involution interchanging $m^\pm$ and $-m^\pm$.

We observe that the Galois automorphisms act on the P-lines and C-lines as follows. The involution $\mathcal{G}^-$ interchanges the two lines in each pair of parallel P-lines, that is,

$$
\mathcal{P}^+ \longrightarrow \mathcal{P}^-.
$$

However it interchanges the two slopes $m^\pm$ so it also interchanges the two line each pair of crossing C-lines.

The involution $\mathcal{G}^+$, on the other hand, interchanges $\sqrt{t+2} \longleftrightarrow -\sqrt{t+2}$ and $m^\pm \longleftrightarrow -m^\mp$ so it takes

$$
\mathcal{C}^+_\pm \longrightarrow \mathcal{C}^-_\mp.
$$

8. The principal double-six

In this section we organize the configuration of lines in terms of a remarkable configuration, due to Schläfli, called a *double-six*.

8.1. **Background.** Recall that a *Schläfli double-six* consists of two ordered sextuples of lines $(a_1, \dots, a_6)$ and $(b_1, \dots, b_6)$ such that:

- If $i \neq j$, then $a_i \;)(\; a_j$.
- If $i \neq j$, then $b_i \;)(\; b_j$.
- $a_i \;)(\; b_j$ if and only if $i = j$.
- a_i and b_j intersect whenever $i \neq j$.

Schläfli's original notation for this is:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

8.2. **Relation with tritangents.** In the last case above, that is, when $i \neq j$, the crossing lines a_i and b_j lie in a tritangent plane which we denote $\mathcal{T}_{ij}$. Write p_{ij} for the point of intersection $a_i \cap b_j$; necessarily $p_{ij} \in \mathcal{T}_{ij}$. Furthermore $\mathcal{T}_{ij}$ meets $\mathcal{S}_k$ in a third line, which we denote c_{ij}.

Proposition 8.2.1. *$i \neq j$, then $c_{ij} = c_{ji}$.*

I am grateful to Damiano Testa for supplying the proof of the following fact:

Lemma 8.2.2. *$i \neq j$, then $c_{ij} = c_{ji}$.*

Proof. Since $a_i \;)(\; a_j$ and $a_j \;)(\; b_j$, it follows (since $p_{ji} \in a_j$ and $p_{ji} \in b_i$) that $p_{ji} \notin a_i \cup b_j$. Since $\mathcal{T}_{ij}$ is a tritangent which intersects S in

$$a_i \cup b_j \cup c_{ij},$$

$p_{ji} \in c_{ij}$. In particular a_j intersects c_{ij}. Why is $p_{ji} \in \mathcal{T}_{ij}$?

Similarly, b_i intersects c_{ij}. Since $a_j, b_i, c_{ij} \in \mathsf{Lines}(S)$, and they mutually cross, $c_{ij} \subset \mathcal{T}_{ji}$. Since these lines are distinct, $c_{ij} = c_{ji}$. $\square$

Although $c_{ij} = c_{ji}$, the tritangent planes are distinct: $\mathcal{T}_{ij} \neq \mathcal{T}_{ji}$. Indeed, the line $c_{ij} = \mathcal{T}_{ij} \cap \mathcal{T}_{ji}$.

Here is a simple example of a double-six:

$$\begin{array}{llllll} a_1 := {}^X\mathcal{C}^{+}_{+}, & a_2 := {}^X\mathcal{C}^{-}_{+}, & a_3 := {}^Y\mathcal{C}^{+}_{+}, & a_4 := {}^Y\mathcal{C}^{-}_{+}, & a_5 := {}^Z\mathcal{C}^{+}_{+}, & a_6 := {}^Z\mathcal{C}^{-}_{+} \\ b_1 := {}^X\mathcal{C}^{-}_{-}, & b_2 := {}^X\mathcal{C}^{+}_{-}, & b_3 := {}^Y\mathcal{C}^{-}_{-}, & b_4 := {}^Y\mathcal{C}^{+}_{-}, & b_5 := {}^Z\mathcal{C}^{-}_{-}, & b_6 := {}^Z\mathcal{C}^{+}_{-} \end{array}$$

The c_{ij} may be computed from Table 2.

The tritangents $\mathcal{T}_{12}, \mathcal{T}_{21}$ are the planes given by the crossing $\mathcal{C}$-lines in the X family. Since they are the parallel planes $x = \pm\sqrt{t+2}$, their intersection is the ideal line ${}^X\mathcal{I}$. Similarly for $\mathcal{T}_{34}, \mathcal{T}_{43}$ (in the Y family), and $\mathcal{T}_{56}, \mathcal{T}_{65}$ (in the Z family).

	b_1	b_2	b_3	b_4	b_5	b_6
a_1		c_{12}	c_{13}	c_{14}	c_{15}	c_{16}
a_2	c_{12}		c_{23}	c_{24}	c_{25}	c_{26}
a_3	c_{13}	c_{23}		c_{34}	c_{25}	c_{36}
a_4	c_{14}	c_{24}	c_{34}		c_{45}	c_{46}
a_5	c_{15}	c_{25}	c_{35}	c_{45}		c_{56}
a_6	c_{16}	c_{26}	c_{36}	c_{46}	c_{56}	

TABLE 1. A Double-Six

	${}^X\mathcal{C}^+_+$	${}^X\mathcal{C}^-_+$	${}^Y\mathcal{C}^+_+$	${}^Y\mathcal{C}^-_+$	${}^Z\mathcal{C}^+_+$	${}^Z\mathcal{C}^-_+$
${}^X\mathcal{C}^-_-$		${}^X\mathcal{I}$	${}^Z\mathcal{P}^+_-$	${}^Z\mathcal{P}^-_+$	${}^Y\mathcal{P}^+_-$	${}^Y\mathcal{P}^-_-$
${}^X\mathcal{C}^+_-$	${}^X\mathcal{I}$		${}^Z\mathcal{P}^-_-$	${}^Z\mathcal{P}^+_+$	${}^Y\mathcal{P}^-_+$	${}^Y\mathcal{P}^+_+$
${}^Y\mathcal{C}^-_-$	${}^Z\mathcal{P}^+_-$	${}^Z\mathcal{P}^-_-$		${}^Y\mathcal{I}$	${}^X\mathcal{P}^+_-$	${}^X\mathcal{P}^-_+$
${}^Y\mathcal{C}^+_-$	${}^Z\mathcal{P}^-_+$	${}^Z\mathcal{P}^+_+$	${}^Y\mathcal{I}$		${}^X\mathcal{P}^-_-$	${}^X\mathcal{P}^+_+$
${}^Z\mathcal{C}^-_-$	${}^Y\mathcal{P}^+_-$	${}^Y\mathcal{P}^-_+$	${}^X\mathcal{P}^+_-$	${}^X\mathcal{P}^-_-$		${}^Z\mathcal{I}$
${}^Z\mathcal{C}^+_-$	${}^Y\mathcal{P}^-_-$	${}^Y\mathcal{P}^+_+$	${}^X\mathcal{P}^-_+$	${}^X\mathcal{P}^+_+$	${}^Z\mathcal{I}$	

TABLE 2. The Principal Double-Six

Next we discuss crossing points and tritangents for a_1, b_3, and a_3, b_1. The other entries in Table 2 will then follow by exploiting symmetry.

Since $a_1 = {}^X\mathcal{C}^+_+$ is defined by $x = \sqrt{t+2}$ and $z = m^+ y$, and $b_3 = {}^Y\mathcal{C}^-_-$ is defined by $y = -\sqrt{t+2}$ and $x = -m^- z$, the crossing point

$$p_{13} = (\sqrt{t+2}, -\sqrt{t+2}, -m^+\sqrt{t+2})$$

Its tangent plane $\mathsf{T}_{p_{13}}$ equals the tritangent $\mathcal{T}_{13}$ which equals, in homogeneous coordinates,

$$\mathcal{T}_{13} = \left[\!\left[1 \quad -1 \quad m^- \quad -\sqrt{t+2}\right]\!\right]$$

which evidently contains the line ${}^Z\mathcal{P}^{+}_{-}$ defined by

$$z = 2, \qquad y = x - \sqrt{t-2}.$$

Since $\mathcal{T}_{13} = \mathsf{T}_{p_{13}}$ contains ${}^Z\mathcal{P}^{+}_{-}$ and $p_{13} \notin {}^Z\mathcal{P}^{+}_{-}$ it follows that $c_{13} = {}^Z\mathcal{P}^{+}_{-}$.

For the pair a_3, b_1, note that the Galois involution $\mathcal{G}^+$ interchanges

$$a_1 = {}^X\mathcal{C}^{+}_{+} \longleftrightarrow b_1 = {}^X\mathcal{C}^{-}_{-}$$

and

$$a_3 = e^Y\mathcal{C}^{+}_{+} \longleftrightarrow b_3 = {}^Y\mathcal{C}^{-}_{-},$$

so

$$p_{31} = \mathcal{G}_- p_{13} = \left(-\sqrt{t+2}, \sqrt{t+2}, -m^-\sqrt{t+2}\right)$$

and

$$\mathcal{T}_{31} = \mathcal{G}_- \mathcal{T}_{13} = \left[\!\left[1 \quad -1 \quad -m^+ \quad \sqrt{t+2}\right]\!\right]$$

Since $\mathcal{G}_-$ fixes ${}^Z\mathcal{P}^{+}_{-}$, the above argument applies and $c_{31} = {}^Z\mathcal{P}^{+}_{-} = c_{13}$.

This argument with $\mathcal{G}_-$ works when this involution is nontrivial, that is, when $\sqrt{t+2} \notin \mathbb{Q}$. However, the conclusion holds by continuity, since $\sqrt{t+2}$ is generically irrational.

9. Clebsch's Diagonal Cubic Surface

When $t = 18$, $\overline{\mathcal{S}_t}$ is Clebsch's *Diagonal Cubic Surface,* the intersection of the cubic hypersurface in $\mathbb{C}\mathsf{P}^3$

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

with the hyperplane $\mathsf{P}^2_1 \subset \mathsf{P}^3$ defined by

$$U_0 + U_1 + U_2 + U_3 + U_4 \;=\; 0.$$

(Compare §4.2.) It clearly enjoys $\mathfrak{S}_5$-symmetry, which is one of the maximal symmetry groups of a smooth projective cubic surface ($\#\mathfrak{S}_5 = 5! = 120$).

In my 2003 publication [9], I noted the level set $\mathcal{S}_t$ for $t = 18$ represented a *dynamical bifurcation:* for $2 \le t \le 18$, the Γ-action is ergodic but for $t > 18$ the action is *not ergodic.* The reason can be traced to the geometric significance of the orbit of the finite Eckard points for $t = 18$, and for $t > 18$ the *Eckard tritangents.*

9.1. **Eckard points.** It has 10 Eckardt points. In addition to the 6 ideal Eckardt points, there are 4 non-ideal Eckardt points $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ described as follows:

$$\begin{aligned}
\mathbf{e}_0 &= (-2,-2,-2) = {}^X\mathcal{P}^-_- \cap {}^Y\mathcal{P}^-_- \cap {}^Z\mathcal{P}^-_- \\
\mathbf{e}_1 &= (-2,2,2) \quad\ = {}^X\mathcal{P}^-_+ \cap {}^Y\mathcal{P}^+_- \cap {}^Z\mathcal{P}^+_+ \\
\mathbf{e}_2 &= (2,-2,2) \quad\ = {}^X\mathcal{P}^+_+ \cap {}^Y\mathcal{P}^+_- \cap {}^Z\mathcal{P}^+_- \\
\mathbf{e}_3 &= (2,2,-2) \quad\ = {}^X\mathcal{P}^+_- \cap {}^Y\mathcal{P}^+_+ \cap {}^Z\mathcal{P}^-_+
\end{aligned}$$

If $p \in \mathcal{S}_t$ which is the intersection of two lines $l_1, l_2 \subset \mathcal{S}_t$ the intersection of $\mathcal{S}_t$ with its tangent plane T_p is a cubic curve in T_p containing $l_1 \cup l_2$. Thus a third line l_3 exists with

$$\mathcal{S}_t \cap \mathsf{T}_p = l_1 \cup l_2 \cup l_3$$

and T_p is a tritangent. In particular, for any Eckardt point p, the tangent plane T_p is a tritangent.

In general tritangents fall into two types: generic tritangents (which contain three lines in general position) and tangent planes at Eckardt points. We call the latter *Eckard tritangents.*

9.2. **Inversion.** The transposition $U_0 \leftrightarrow U_4$ corresponds to the involution

$$(x,y,z) \xleftrightarrow{(04)} \left(\frac{x}{x+y+z-2}, \frac{y}{x+y+z-2}, \frac{z}{x+y+z-2} \right)$$

which we call *inversion.* It maps the ideal tritangent plane $\mathcal{T}_\infty = \mathsf{Syl}_0$ to the Sylvester tritangent Syl_4 whose affine piece is defined by:

$$x + y + z = 2,$$

the plane containing the three Eckardt points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

This *inversion* which fixes the *Eckard tritangent* $\mathsf{T}_{\mathsf{Eck}_0}$ whose affine piece is defined by

$$x + y + z = 6$$

. The Sylvester tritangent Syl_4 and the Eckard tritangent $\mathsf{T}_{\mathsf{Eck}_0}$ are related by the Galois involution $\mathcal{G}^-$.

The subgroup $\mathfrak{S}_4 < \mathfrak{S}_5$ consists of the linear automorphisms discussed earlier, and $\mathfrak{S}_5 = \langle \mathfrak{S}_4, (04) \rangle$.

9.3. **Dynamical significance.** The tritangent $\mathcal{T}_{\mathbf{e}_0} = \mathsf{T}_{\mathbf{e}_0}\mathcal{S}_{18}$ defined by:

$$x + y + z + (2 + \sqrt{t-2}) = 0$$

containing the three lines:

$$\begin{aligned} c_{23} = {}^{Z}\mathcal{P}^{-}_{+} &:= \{z = -2, \qquad x + y + \sqrt{t-2} = 0\} \\ c_{46} = {}^{X}\mathcal{P}^{-}_{+} &:= \{x = -2, \qquad y + z + \sqrt{t-2} = 0\} \\ c_{15} = {}^{Y}\mathcal{P}^{-}_{+} &:= \{y = -2, \qquad z + x + \sqrt{t-2} = 0\} \end{aligned}$$

is dynamically interesting, for $k \geq 18$. The orthant Ω defined by $x, y, z \leq -2$ parametrizes the Fricke space of the 3-holed sphere with the *standard marking,* the one whose generators correspond to the boundary components. It meets $\mathcal{S}_k$ in a wandering domain for the action of

$$\Gamma := \mathsf{Out}(\mathsf{F}_2) \cong \pi_0\big(\mathsf{Homeo}(S)\big) \cong \mathsf{GL}(2, \mathbb{Z})$$

and is bounded by T_{13}. Geometrically, points in the orbit $\Gamma\Omega$ correspond to homotopy equivalences $S \rightsquigarrow M$, where M is a complete hyperbolic surface homeomorphic to a three-holed sphere.

The Eckardt point $\mathbf{e}_0 := (-2, -2, -2)$ in the Clebsch cubic $\mathcal{S}_{18}$ corresponds to the complete finite area 3-*punctured* sphere M. This Eckardt point arises as the domain Ω collapses as $k \searrow 18$.) The four $\mathcal{P}$-lines in each coordinate family bound an open annulus, whose levels are ellipses. The corresponding cyclic group of Dehn twists acts minimally (and ergodically) on almost every level ellipse. This leads to chaotic dynamics (ergodicity with respect to the Poisson measure arising from the invariant function κ and Euclidean volume form) on the complement of the orbit $\Gamma \cdot \Omega$ of the wandering domain.

10. Enumerating tritangents

A general cubic surface contains 45 tritangent planes. We can account for them as follows on the Clebsch cubic $\overline{\mathcal{S}_{18}}$, which has 4 finite Eckardt points.

First there is the ideal tritangent $\mathcal{T}_\infty$ containing all three ideal lines.

There are 30 tritangents arising from the double-six matrix. Namely the 12 s partition into the two sextuples $a_1, \dots, a_6$ and $b_1, \dots, b_6$. Whenever $i \neq j$, the lines a_i, b_j extend to a tritangent also containing c_{ij}. The 15 lines c_{ij} consist of all 12 of the P-lines and all 3 of the ideal lines. The ideal lines fall into the tritangents containing a pair of crossing C-lines

Corresponding to 6 ideal Eckardt points (the common ideal points of a parallel pair of P-lines) are 6 tritangents.

Corresponding to each of the 4 finite Eckardt points is tritangent of concurrent P-lines. For example §9.3 discusses the Eckardt point

$$\mathbf{e}_0 := c_{23} \cap c_{46} \cap c_{15} = {}^{Z}\mathcal{P}^{-}_{-} \cap {}^{X}\mathcal{P}^{-}_{-} \cap {}^{Y}\mathcal{P}^{-}_{-}.$$

whose tangent plane $\mathsf{T}_{\mathbf{e}_0}\mathcal{S}_{18}$ is the tritangent defined by

$$x + y + z + (2 + \sqrt{t-2}) = 0$$

Its image under the Galois involution $\mathcal{G}^-$ is another tritangent containing lines

$${}^{Z}\mathcal{P}^-_- \cap {}^{X}\mathcal{P}^-_- \cap {}^{Y}\mathcal{P}^-_-.$$

and defined by

$$x + y + z + (2 - \sqrt{t-2}) = 0.$$

Thus the 4 tangent planes to E-points and their Galois conjugates give a total of 8 tritangents.

This accounts for all $1+30+6+8 = 45$ tritangents to $\overline{\mathcal{S}_t}$ in the special case $k = 18$.

11. A Steiner trihedral pair

Let $S \subset \mathsf{P}^3$ be a smooth projective cubic. A *trihedral pair* is an unordered pair

$$\big\{\{\phi_1,\phi_2,\phi_3\},\{\phi_4,\phi_5,\phi_6\}\big\}$$

of unordered triples of covectors

$$\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6 \ \in \mathsf{V}^*$$

such that

$$(\phi_1\phi_2\psi_3) + (\phi_4\phi_5\phi_6) = 0$$

defines S. For example,

$$\begin{aligned} \phi_1 &:= X + 2W & \qquad \phi_4 &:= W \\ \phi_2 &:= Y + 2W & \qquad \phi_5 &:= X + Y + Z + (2 + \sqrt{t-2})W \\ \phi_3 &:= Z + 2W & \qquad \phi_6 &:= X + Y + Z + (2 - \sqrt{t-2})W \end{aligned}$$

is a trihedral pair for our family

$$S = \overline{\mathcal{S}_t} := \Big\{(X^2+Y^2+Z^2)W - XYZ - (t+2)W^3 = 0\Big\}.$$

The corresponding six tritangents are:

$$\begin{aligned} P_1 &:= \{x = -2\} & \qquad P_4 &= \mathcal{T}_\infty \\ P_2 &:= \{y = -2\} & \qquad P_5 &= \Big\{x + y + z = -2 - \sqrt{t-2}\Big\} \\ P_3 &:= \{z = -2\} & \qquad P_6 &= \Big\{x + y + z = -2 + \sqrt{t-2}\Big\} \end{aligned}$$

given for the affine planes, except for the ideal plane $P_4 = \mathcal{T}_\infty$.

Notation

Ideal lines: ${}^X\mathcal{I}$, ${}^Y\mathcal{I}$, ${}^Z\mathcal{I}$
Coordinate ideal points: ${}^X\mathbf{p}_\infty, {}^Y\mathbf{p}_\infty, {}^Z\mathbf{p}_\infty$
$\mathcal{C}$-lines: ${}^X\mathcal{C}^\pm_\pm$, ${}^Y\mathcal{C}^\pm_\pm$, ${}^Z\mathcal{C}^\pm_\pm$
$\mathcal{P}$-lines: ${}^X\mathcal{P}^\pm_\pm$, ${}^Y\mathcal{P}^\pm_\pm$, ${}^Z\mathcal{P}^\pm_\pm$
Ideal tritangent plane: $\mathfrak{T}_\infty$
Ideal Eckardt points: $\mathsf{Eck}_\infty X\pm, \mathsf{Eck}_\infty Y\pm, \mathsf{Eck}_\infty Z\pm$
Finite Eckardt points: $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Symmetric groups: $\mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$
Sylvester planes: $\mathsf{Syl}_0, \mathsf{Syl}_1, \mathsf{Syl}_2, \mathsf{Syl}_3, \mathsf{Syl}_4$
Ends: $\mathsf{End}_0, \mathsf{End}_1, \mathsf{End}_2, \mathsf{End}_3$
Critical points of κ: $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{o}$
Dihedral characters: ${}^X\mathsf{Dih}^\pm, {}^Y\mathsf{Dih}^\pm, {}^Z\mathsf{Dih}^\pm$
Sign-changes $\delta_1, \delta_2, \delta_3, \mathbf{1}$ comprise Δ, realizing $\big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big) \triangleleft \mathfrak{S}_4$
Vieta involutions ${}^X\nu$, ${}^Y\nu$, ${}^Z\nu$

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