Hyperbolizing Surfaces

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- The *mapping class group* $\mathrm{Mod}(\Sigma) \cong \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ acts on $\mathrm{Hom}(\pi, G)/G$.
- Representations $\pi \longrightarrow G$ arise from *locally homogeneous geometric structures* on Σ, modelled on homogeneous spaces of G.

Navigating the deformation space

- The fundamental group $\pi = \pi_1(\Sigma)$ is the fundamental group of a closed orientable surface admits a presentation

$$\pi = \langle A_1, \dots, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

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- Associated to simple closed curves $\alpha \subset \Sigma$ are *generalized twist deformations*, paths in $\mathrm{Hom}(\pi, G)$ *supported* on α.
- For example, if α is the nonseparating simple loop A_1:

$$\rho_t : \begin{cases} A_i & \longmapsto \rho(A_i) \text{ if } i \geq 1 \\ B_j & \longmapsto \rho(B_j) \text{ if } j > 1 \\ B_1 & \longmapsto \rho(B_1)\zeta(t) \end{cases}$$

where $\zeta(t)$ is a path in the centralizer of $\rho(A_1)$.

Generalized twist flows

- Similarly if $C = [A_1, B_1] \dots [A_k, B_k]$ corresponds to a *separating simple loop* on Σ, then

$$\rho_t : \begin{cases} A_i & \longmapsto \rho(A_i) \text{ if } i \leq k \\ B_i & \longmapsto \rho(B_i) \text{ if } i \leq k \\ A_i & \longmapsto \zeta(t)\rho(A_i)\zeta(t)^{-1} \text{ if } i > k \\ B_i & \longmapsto \zeta(t)\rho(B_i)\zeta(t)^{-1} \text{ if } i > k \end{cases}$$

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  where $\zeta(t)$ is a path in the centralizer of $\rho(C)$.
- Example for $G = \mathrm{SL}(2,\mathbb{R})$: When $\rho(A_1)$ leaves invariant a geodesic $l \subset \mathrm{H}^2$, then $\zeta(t)$ is a group of *transvections* along l.

Observing the deformation space

- A natural class of functions on $\mathrm{Hom}(\pi, G)/G$ arise from functions $G \xrightarrow{f} \mathbb{R}$ invariant under conjugation and $\alpha \in \pi$:

$$\begin{aligned} \mathrm{Hom}(\pi, G)/G &\xrightarrow{f_\alpha} \mathbb{R} \\ [\rho] &\longmapsto f\big(\rho(\gamma)\big) \end{aligned}$$

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- The *trace* of any linear representation $G \longrightarrow \mathrm{GL}(N,\mathbb{R})$
- The *geodesic displacement function* (only defined for hyperbolic elements)

$$\mathrm{tr}(\gamma) = \pm 2\cosh\big(\ell(\gamma)/2\big)$$

if $\gamma \in \mathrm{SL}(2,\mathbb{R})$ is hyperbolic.

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- When α is represented by a simple closed curve, and $G \xrightarrow{f} \mathbb{R}$ is an invariant function, then the Hamiltonian flow of f_α is covered by a *generalized twist flow* on $\mathrm{Hom}(\pi, G)$.

Fricke space

- The deformation space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures Σ identifies with the space of embeddings

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- Components of $\mathsf{Hom}(\pi, \mathsf{PSL}(2,\mathbb{R}))$ are detected by the *Euler class* of the associated oriented $\mathbb{RP}^1$-bundle over Σ:

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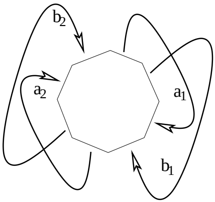
- Equality $\iff \rho$ is a discrete embedding. (1980)

Branched hyperbolic structures

- Obtain a genus g surface from a $4g$-gon.



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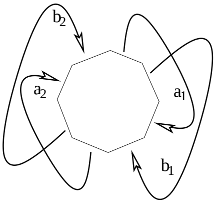

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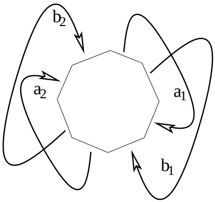
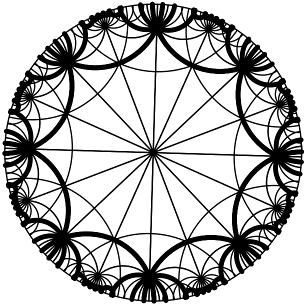
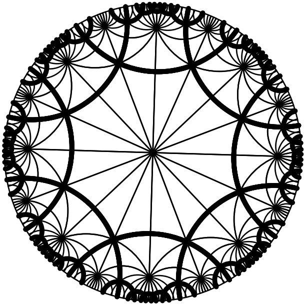


- If the sum of the interior angles is $2\pi k$, where $k \in \mathbb{Z}$, then quotient space is a hyperbolic surface with one singularity (the image of the vertex) with cone angle $2\pi k$.
- The holonomy representation of a hyperbolic surface with cone angles $2\pi k_i$ extends to $\pi_1(\Sigma)$ with Euler number

$$e(\rho) = 2 - 2g + \sum k_i.$$

A hyperbolic surface of genus two

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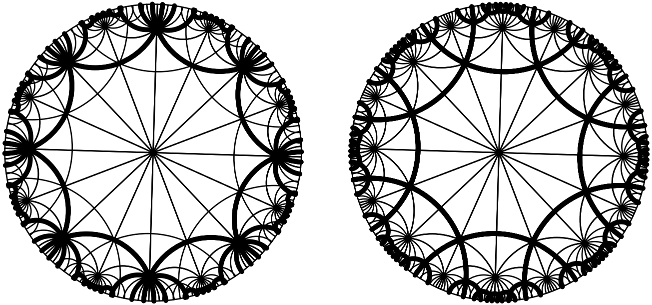

- Identifying a regular octagon with angles $\pi/4$ yields a nonsingular hyperbolic surface with $e(\rho) = \chi(\Sigma) = -2$.

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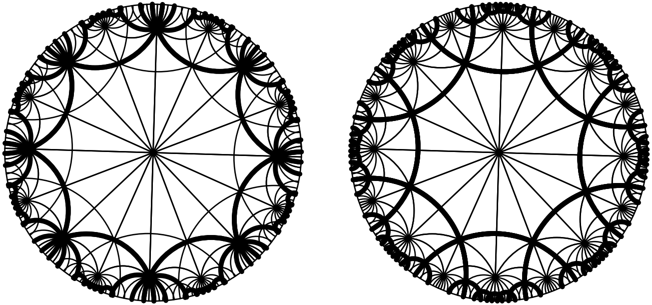

- Identifying a regular octagon with angles $\pi/4$ yields a nonsingular hyperbolic surface with $e(\rho) = \chi(\Sigma) = -2$.
- But when the angles are $\pi/2$, the surface has one singularity with cone angle 4π and

$$e(\rho) = 1 + \chi(\Sigma) = -1.$$

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- *Conjecture:* every representation with dense image occurs as the holonomy of a branched hyperbolic structure.

Dynamic/homotopic triviality

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- Decomposing Σ into pants along curves $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ where $N = 3g - 3$, the *Fenchel-Nielsen mapping*

$$\mathfrak{F}(\Sigma) \xrightarrow{\ell_\Gamma} (\mathbb{R}_+)^N$$
$$\langle M \rangle \longmapsto \big(\ell_1(M), \dots \ell_N(M)\big)$$

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- ℓ_Γ moment map for *completely integrable Hamiltonian system.* (Wolpert 1983)

Canonical coordinates

- Choose a section of ℓ_Γ to define *twist coordinates* $\tau_1, \dots, \tau_N$, to trivialize the principal bundle:

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- The symplectic form equals

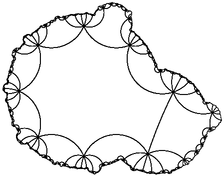
$$\sum_{i=1}^{N} d\ell_i \wedge d\tau_i.$$

(Wolpert 1985)

Quasi-Fuchsian groups

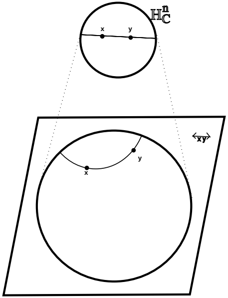
The group of orientation-preserving isometries of $H^3_{\mathbb{R}}$ equals $PSL(2,\mathbb{C})$. Close to Fuchsian representations in $PSL(2,\mathbb{R})$ are *quasi-Fuchsian representations.*

- Quasi-fuchsian representations are discrete embeddings.
- Quasi-fuchsian representations comprise a cell $\mathcal{QF}$ upon which $\text{Mod}(\Sigma)$ acts properly.
- $\text{Hom}(\pi, SL(2,\mathbb{C}))$ is connected, and the closure of $\mathcal{QF}$ consists of all discrete embeddings.
- The discrete embeddings are *not open* and do not comprise a component of $\text{Hom}(\pi, G)/G$.



Complex hyperbolic geometry

- Complex hyperbolic space $H^n_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^n$ with the *Bergman metric* invariant under the projective transformations in $\mathbb{CP}^n$.



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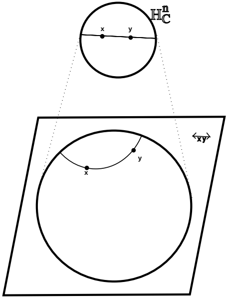

- $\mathbb{C}$- linear subspaces meet $H^n_{\mathbb{C}}$ in totally geodesic subspaces.

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- Generalized to *maximal representations* by Burger-Iozzi-Wienhard and Bradlow-Garcia-Prada-Gothen-Mundet. ($\mathrm{Mod}(\Sigma)$ acts properly of maximal components, well-displacing property, determination of topological type...)

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- For an $\mathrm{SU}(1,1)$-representation ρ_0, the neighborhood of

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  in $\mathrm{Hom}(\pi, \mathrm{SU}(2,1))$ looks like the product of $\mathrm{Hom}(\pi, \mathrm{U}(1,1) \times \mathrm{U}(1))$ and a cone defined by a quadratic form of signature $e(\rho_0)$ on $\mathbb{R}^{4g-4}$.
- For all even e with $|e| \leq 2g-2$, the corresponding component of $\mathrm{Hom}(\pi, \mathrm{SU}(2,1))$ contains *discrete embeddings.* (G Kapovich Leeb 2001)

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- Are all algebraic limits strong?
- Do degenerate surface groups exist?

Real projective geometry

- A *marked convex* $\mathbb{RP}^2$*-structures* is a diffeomorphism

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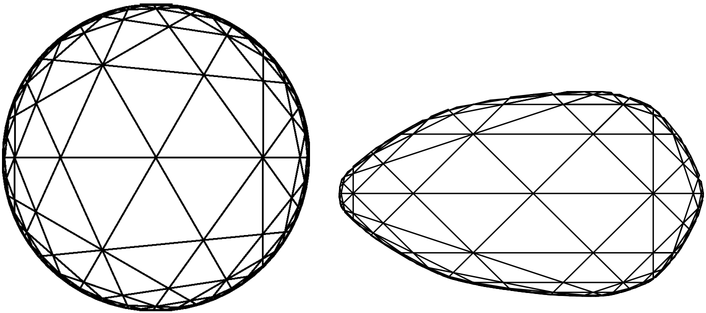
- Every $\mathbb{RP}^2$-structure canonically decomposes as a union of convex structures (Choi 1988).

Deformations of triangle groups



Domains in $\mathbb{RP}^2$ tiled by $(3, 3, 4)$-triangles.

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- $\mathfrak{C}(\Sigma)$ identifies with the holomorphic vector bundle over Teich(Σ) whose fiber over a marked Riemann surface X equals the vector space $H^0(X, (\kappa_X)^2)$ of *holomorphic cubic differentials*
  (Labourie 1997, Loftin 2001).

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- Generalizes *shearing coordinates.* (Penner 1987)

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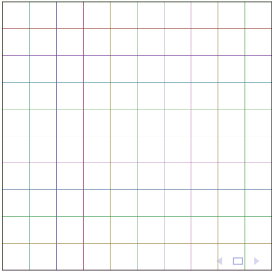

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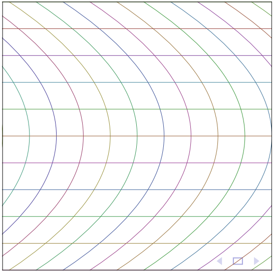

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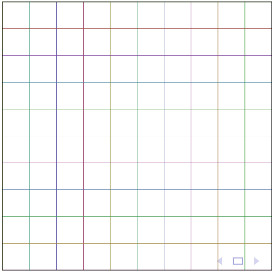

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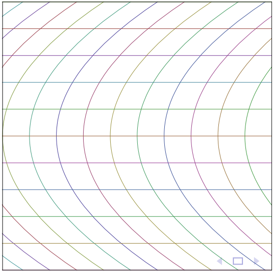

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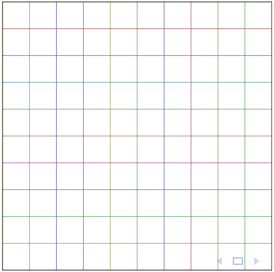

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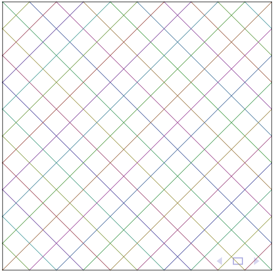

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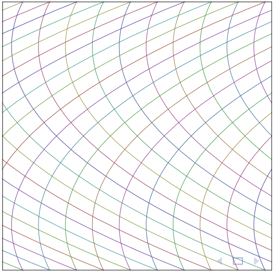

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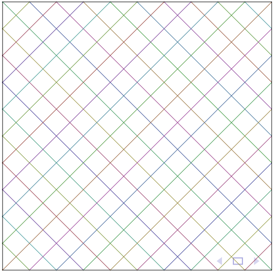

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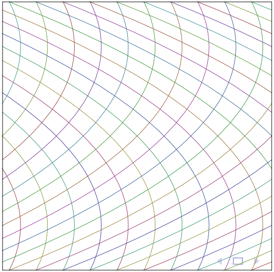

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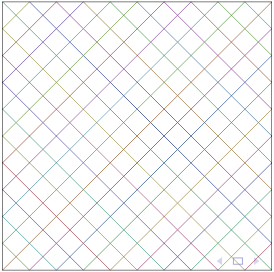

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- Contrast to the *proper* action of $\text{Mod}(\Sigma) \cong PGL(2,\mathbb{Z})$ on $\mathfrak{F}(\Sigma)$ by *projective* transformations.