

CROOKED SURFACES AND ANTI-DE SITTER GEOMETRY

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Dedicated to the memory of Shoshichi Kobayashi

ABSTRACT. Crooked planes were defined by Drumm to bound fundamental polyhedra in Minkowski space for Margulis spacetimes. They were extended by Frances to closed polyhedral surfaces in the conformal compactification of Minkowski space (Einstein space) which we call crooked surfaces. The conformal model of anti-de Sitter space is the interior of the quotient of Einstein space by an involution fixing an Einstein plane. The purpose of this note is to show that the crooked planes defined in anti-de Sitter space recently by Danciger-Guéritaud-Kassel lift to restrictions of crooked surfaces in Einstein space which are adapted under the involution of Einstein space defining anti-de Sitter space.

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1. INTRODUCTION

In 1990, Todd Drumm [8] introduced crooked planes to build fundamental polyhedra for free discrete groups acting properly and isometrically on 3-dimensional Minkowski space \mathbb{E}^3 . Minkowski space is the Lorentzian analog of Euclidean space, and anti-de Sitter space is the Lorentzian analog of hyperbolic space. Recently Jeff Danciger, François Guéritaud and Fanny Kassel [6] introduced analogous surfaces in 3-dimensional anti-de Sitter space AdS^3 .

The conformal compactification of Minkowski space is the *Einstein universe* Ein^3 , the geometry of which has been extensively studied by Charles Frances [11]. In particular Frances [12] considered the extensions of crooked planes to Ein^3 , which he called *generalized crooked planes*. (See [2] for an expanded treatment of Einstein geometry and generalized crooked planes, which we renamed *crooked surfaces*.)

The purpose of this note is to interpret the AdS -crooked planes of Danciger-Guéritaud-Kassel in terms of crooked surfaces in Ein^3 .

Here is a precise statement of our main result. 3-dimensional Minkowski space \mathbb{E}^3 embeds in the 3-dimensional Einstein universe Ein^3 as the complement of a lightcone. The unique nontrivial double covering $\widehat{\text{AdS}}$ of 3-dimensional anti-de Sitter space AdS^3 embeds in Ein^3 as the complement of a 2-dimensional Einstein hypersphere $\text{Ein}^2 \subset \text{Ein}^3$. The conformal involution $I_{\mathcal{C}}$ of Ein^3 whose fixed set equals Ein^2 realizes the deck transformation of the complement

$$\text{Ein}^3 \setminus \text{Ein}^2 = \widehat{\text{AdS}}.$$

Let $p \in \text{AdS}^3$, and let $\hat{p}, -\hat{p} \in \widehat{\text{AdS}}$ be its two lifts under the double covering $\widehat{\text{AdS}} \rightarrow \text{AdS}^3$. Let $\mathbf{s} \in T_p(\text{AdS}^3)$ be a unit-spacelike vector, and abusing notation, denote the corresponding tangent vector in $T_{\hat{p}}(\widehat{\text{AdS}})$ by \mathbf{s} as well. Denote the origin in $T_p(\text{AdS}^3)$ by $\mathbf{0}_p$. The *tangent cone of $\mathcal{C}(p, \mathbf{s})$ at p* is the subset of $T_p(\text{AdS}^3)$ consisting of tangent vectors to smooth paths in \mathcal{C} at p . Identify the tangent space $T_p(\text{AdS}^3)$ — as a Lorentzian vector space — with Minkowski space \mathbb{E}^3 .

A crooked surface in Ein^3 is determined by an ordered quadruple $(p_0, p_\infty, p_1, p_2) \in \text{Ein}^3$ where p_0, p_∞ are non-incident and p_i is incident to both p_0, p_∞ for $i = 1, 2$. (Compare [2, 16, 4].) Say that the crooked surface $\mathfrak{C}(p_0, p_\infty, p_1, p_2)$ is *$I_{\mathcal{C}}$ -adapted* if and only if $I_{\mathcal{C}}$ fixes p_1 and p_2 and interchanges p_0 and p_∞ . This is somewhat stronger than the condition that *the set $\mathfrak{C}(p_0, p_\infty, p_1, p_2)$ is $I_{\mathcal{C}}$ -invariant*, and rules out potentially interesting hypersurfaces which resemble crooked planes, but

are not crooked planes. A characterization of this condition in terms of the *spine* of a crooked surface is given in §4.

Main Theorem. (1) *Let $p \in \text{AdS}^3$ and $\mathbf{s} \in T_p(\text{AdS}^3)$ be unit-spacelike as above. Then the tangent cone to the AdS-crooked plane $\mathcal{C}(p, \mathbf{s})$ is a crooked plane $\mathcal{C}(\mathbf{0}_p, \mathbf{s}) \subset \mathbb{E}^3 \approx T_p(\text{AdS}^3)$ and*

$$\mathcal{C}(p, \mathbf{s}) = \text{Exp}_p(\mathcal{C}(\mathbf{0}_p, \mathbf{s})).$$

(2) *Let $\widehat{\mathcal{C}}(p, \mathbf{s})$ be the inverse image of $\mathcal{C}(p, \mathbf{s})$ under the double covering $\widehat{\text{AdS}} \rightarrow \text{AdS}^3$, and $\mathfrak{C}(p, \mathbf{s})$ be the closure of the image of the embedding*

$$\widehat{\mathcal{C}}(p, \mathbf{s}) \hookrightarrow \widehat{\text{AdS}} \xrightarrow{\Psi} \text{Ein}^3.$$

Then $\mathfrak{C}(p, \mathbf{s})$ is an $I_{\mathcal{I}}$ -adapted crooked surface in Ein^3 .

(3) *Conversely suppose that \mathfrak{C} is an $I_{\mathcal{I}}$ -adapted crooked surface in Ein^3 . Then \mathfrak{C} arises from an AdS-crooked plane by the above construction.*

The first section of the paper develops anti-de Sitter geometry, defines AdS-crooked planes, describes the double cover, and proves (1). The second section discusses the embedding Ψ of the double cover $\widehat{\text{AdS}}$ in Ein^3 and the conformal model of anti-de Sitter space. Using this conformal realization and its symmetries, the final section proves (2) by computing one example explicitly. Symmetry is exploited one last time to prove (3), the converse statement in the Main Theorem.

Our viewpoint follows the spirit of classical projective geometry, but in the *conformal* context. Conformal models for hyperbolic and elliptic geometry arise from imposing involutions on the sphere S^n , the model space for conformal (Riemannian) geometry. The conformal model of \mathbb{H}^n consists of the two hemispheres of S^n complementary to the equator $S^{n-1} \hookrightarrow S^n$, each given the Poincaré metric. Reflection in the equator is the involution ι *defining hyperbolic geometry*, in the following sense. Hyperbolic space \mathbb{H}^n is the quotient of the complement of $\text{Fix}(\iota) = S^{n-1}$ by the cyclic group $\langle \iota \rangle$. The quotient map

$$S^n \setminus \text{Fix}(\iota) \longrightarrow (S^n \setminus \text{Fix}(\iota)) / \langle \iota \rangle =: \mathbb{H}^n$$

is a trivial covering space. Either hemisphere is a fundamental domain for the action of $\langle \iota \rangle$.

Here anti-de Sitter geometry arises from conformal Lorentzian geometry (on the Einstein universe Ein^3) by imposing an involution. Just as the complement of the fixed-point set of conformal inversion on S^n is a double covering space of hyperbolic space, the complement of the fixed set of the Lorentzian conformal inversion is a double covering space of

anti-de Sitter space. Unlike the Riemannian case above, this covering space is nontrivial.

Metric bisectors in real hyperbolic space \mathbf{H}^n are totally geodesic hypersurfaces. In this model, totally geodesic hypersurfaces are hyperspheres orthogonal to the equator. Orthogonality to the equator is equivalent to invariance under the reflection in the equator, the *involution defining* \mathbf{H}^n . In this way our observation is a direct analog in the crooked context.

A similar viewpoint was adopted in [14] to develop the theory of metric bisectors in complex hyperbolic space in terms of real analytic hypersurfaces in complex projective space $\mathbf{P}_{\mathbb{C}}^n$ which we called *extors*. Metric bisectors (or equidistant hypersurfaces) in $\mathbf{H}_{\mathbb{C}}^n$ analytically continue to extors in $\mathbf{P}_{\mathbb{C}}^n$. Furthermore they can be characterized as those extors adapted to the anti-polarity defining $\mathbf{H}_{\mathbb{C}}^n \subset \mathbf{P}_{\mathbb{C}}^n$. Metric bisectors in *complex elliptic space* ($\mathbf{P}_{\mathbb{C}}^n$ with the Fubini-Study metric) can be similarly characterized as extors in $\mathbf{P}_{\mathbb{C}}^n$ which are adapted to the anti-polarity defining complex elliptic geometry.

A consequence of our observation is that the disjointness criterion for crooked surfaces in Ein^3 proved by Charette, Francoeur, and Lareau-Dussault [4] specializes to a disjointness criterion for the AdS-crooked planes defined by Danciger, Guéritaud, and Kassel. (This extends the disjointness criteria for crooked planes developed in Buelle, Charette, Drumm and Goldman [3] and [9]).

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Notation and Terminology. Homogeneous coordinates on projective space will be denoted $[X : Y : Z : U : V]$, while inhomogeneous coordinates on affine Minkowski space will be denoted (x, y, z) . As we predominantly consider Lorentzian 3-manifolds, with two spatial dimensions and one temporal dimension, we shall refer to the *ambient* dimension: Thus we denote Minkowski space $\mathbb{E}^{2,1}$ by \mathbb{E}^3 , anti-de Sitter space $\text{AdS}^{2,1}$ by AdS^3 and the Einstein universe $\widehat{\text{Ein}}^{2,1}$ by Ein^3 , or simply Ein . Denote the double cover of AdS^3 by $\widehat{\text{AdS}}$ and the 2-dimensional Einstein universe (one spatial and one temporal dimension) by Ein^2 .

Denote a symmetric space by $X = G/H$, where G is the isometry group and H is the isotropy group. Denote the exponential map of a complete pseudo-Riemannian manifold M at a point $p \in M$ by

$$T_p(M) \xrightarrow{\text{Exp}_p} M$$

and the exponential map for a Lie group G with Lie algebra \mathfrak{g} by

$$\mathfrak{g} \xrightarrow{\text{exp}} G.$$

Denote the adjoint representation of a Lie group on its Lie algebra by

$$G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{g}).$$

Denote the vector space of $n \times n$ real matrices by $\text{Mat}_n(\mathbb{R})$ and the zero and identity matrices by $\mathbf{0}$ and $\mathbf{1}$, respectively. Denote the transpose of a matrix M by M^\dagger . Denote equivalence classes in $\text{PSL}(2, \mathbb{R})$ of matrices $A \in \text{SL}(2, \mathbb{R})$ by $\pm A$. Denote the Lie algebra of $\text{SL}(2, \mathbb{R})$, consisting of traceless 2×2 real matrices, by $\mathfrak{sl}(2, \mathbb{R})$, and the hyperbolic plane (in the Poincaré upper halfplane model) by \mathbb{H}^2 . Denote the geodesic in \mathbb{H}^2 corresponding to the positive imaginary axis by \mathbb{H}^1 . Denote the closure of a subset S by \bar{S} and lifts of points, subsets S , etc. from a space M to a double covering space $\widehat{M} \rightarrow M$ by \widehat{S} . Denote the set of points fixed by a transformation f by $\text{Fix}(f)$. Denote the cyclic group generated by f by $\langle f \rangle$.

2. ANTI-DE SITTER GEOMETRY

We begin with a review of the geometry of AdS , using the $\text{PSL}(2, \mathbb{R})$ -model. See Danciger [5], §4.8, for an expanded treatment of this model and its projective realization.

2.1. Isometries. A *Lorentzian symmetric space* is a Lorentzian manifold X such that for every point p , the isometry $-\mathbf{1}$ of the tangent space $T_p X$ is the differential of a (necessarily unique) isometry (denoted ι_p) of X . We call ι_p the *symmetry* of X at p . The isometry group acts

transitively. For the general theory of (non-Riemannian) symmetric spaces, see Eschenburg [10], Helgason [15], and Wolf [20].

2.1.1. *The $\mathrm{PSL}(2, \mathbb{R})$ model.* A convenient model for 3-dimensional anti-de Sitter geometry is the group $\mathrm{PSL}(2, \mathbb{R})$. Its Lorentzian structure is bi-invariant, and arises from an Ad -invariant inner product on its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. In this case AdS^3 corresponds to $\mathrm{PSL}(2, \mathbb{R})$ and the identity component of its isometry group is

$$G^0 := \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$$

acting faithfully by *left- and right-multiplication* on $\mathrm{PSL}(2, \mathbb{R})$:

$$(1) \quad (g_1, g_2) : x \longmapsto g_1 x g_2^{-1}$$

The most convenient basepoint in a group is of course its identity element e . The isotropy group at e is the subgroup $H \subset G^0$ corresponding to the diagonal $g_1 = g_2$. Furthermore $H \cong \mathrm{PSL}(2, \mathbb{R})$ acting by *inner automorphisms* of $\mathrm{PSL}(2, \mathbb{R})$.

2.1.2. *Components of the isometry group.* Isometries not in the identity component are described, for example, in Danciger [5], §4.8 (p.74). Briefly, replace $\mathrm{PSL}(2, \mathbb{R})$ by the group $\mathrm{PGL}(2, \mathbb{R})$ of *all* isometries of \mathbb{H}^2 , including those which reverse orientation on \mathbb{H}^2 . The identity component of $\mathrm{PGL}(2, \mathbb{R})$ equals $\mathrm{PSL}(2, \mathbb{R})$. The component consisting of orientation-reversing isometries is the coset

$$R \cdot \mathrm{PSL}(2, \mathbb{R}) = \mathrm{PSL}(2, \mathbb{R}) \cdot R \subset \mathrm{PGL}(2, \mathbb{R})$$

where, for example,

$$R := \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{PGL}(2, \mathbb{R})$$

corresponds to a reflection in a geodesic $\mathbb{H}^1 := \mathrm{Fix}(R) \subset \mathbb{H}^2$ (the imaginary axis in the upper halfplane model of \mathbb{H}^2).

Now consider the action of $\mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ by left- and right-multiplication on $\mathrm{PGL}(2, \mathbb{R})$ defined by (1). Then the group G of all orientation-preserving isometries of AdS^3 equals the index-two subgroup of $\mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ which stabilizes the component $\mathrm{PSL}(2, \mathbb{R})$ of $\mathrm{PGL}(2, \mathbb{R})$. The orientation-preserving isometry group G contains two components; the identity component is $G^0 = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. The new component is the coset

$$(R, R) \cdot G^0 = G^0 \cdot (R, R) \subset G.$$

This coset consists of isometries which preserve the *ambient orientation* of AdS^3 but reverse *time-orientation*.

This describes an isomorphism $G \xrightarrow{\cong} \mathrm{SO}(2, 2)$. Transformations of the form (Rg_1, g_2) or (g_1, Rg_2) , where $g_1, g_2 \in \mathrm{PSL}(2, \mathbb{R})$, do *not* act orthogonally, but rather take the quadratic form to its negative.

2.1.3. *Symmetries in points.* The *symmetry* of the symmetric space AdS^3 at e is given by group-inversion:

$$(2) \quad \begin{array}{c} \mathrm{PSL}(2, \mathbb{R}) \xrightarrow{\iota_e} \mathrm{PSL}(2, \mathbb{R}) \\ g \longmapsto g^{-1}. \end{array}$$

It is equivariant respecting the permutation automorphism of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$:

$$\begin{array}{c} \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \longrightarrow \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \\ (g_1, g_2) \longmapsto (g_2, g_1) \end{array}$$

because (applying (2) to (1)):

$$(3) \quad \iota_e \circ (g_1, g_2) = (g_2, g_1) \circ \iota_e.$$

The differential of ι_e is the antipodal map $-\mathbf{1}$ on the tangent space $T_e(\mathrm{AdS}^3) \cong \mathfrak{sl}(2, \mathbb{R})$, and it reverses orientation. It generates the other cosets of the full group of isometries of AdS^3 , including isometries which reverse orientation as well as possibly reversing time-orientation.

In this way AdS^3 is a Lorentzian symmetric space with ι_e as the symmetry in e . More generally, the symmetry in the point $g \in \mathrm{AdS}^3 = \mathrm{PSL}(2, \mathbb{R})$ is:

$$(4) \quad \begin{array}{c} \mathrm{AdS}^3 \xrightarrow{\iota_g} \mathrm{AdS}^3 \\ x \longmapsto gx^{-1}g \end{array}$$

2.2. **Geodesics and totally geodesic surfaces.** Geodesics in AdS^3 are left-cosets of one-parameter subgroups in $\mathrm{PSL}(2, \mathbb{R})$. Since

$$g \exp(t\xi) = \exp(t \mathrm{Ad}(g)\xi) g,$$

geodesics correspond to right-cosets as well. AdS^3 is *geodesically complete*: since $\exp(t\xi)$ is defined for all $t \in \mathbb{R}$, every geodesic extends indefinitely in its affine parametrization. Furthermore any two points in AdS^3 are connected by a geodesic.

2.2.1. *Geometry of the tangent space.* Before describing the geodesics and totally geodesic surfaces, we review basic facts about the Lorentzian vector space $\mathfrak{sl}(2, \mathbb{R}) \cong \mathbb{R}^{2,1}$. Let $\mathbb{R}^{2,1}$ denote the 3-dimensional Lorentzian vector space with inner product associated to the quadratic form:

$$\begin{aligned} \mathbb{R}^{2,1} &\longrightarrow \mathbb{R} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\longmapsto x^2 + y^2 - z^2 \end{aligned}$$

Under the identification

$$(5) \quad \begin{aligned} \mathfrak{sl}(2, \mathbb{R}) &\longrightarrow \mathbb{R}^{2,1} \\ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} &\longmapsto \begin{bmatrix} a \\ (b+c)/2 \\ (b-c)/2 \end{bmatrix} \end{aligned}$$

the inner product in $\mathbb{R}^{2,1}$ identifies with the inner product in $\mathfrak{sl}(2, \mathbb{R})$ associated to the quadratic form:

$$(6) \quad \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathbb{R} \\ A \longmapsto -\det(A) = \frac{1}{2}\mathrm{tr}(A^2).$$

We denote the corresponding (Lorentzian) inner product by $A \cdot B$. (The Killing form

$$A \longmapsto \mathrm{tr}(\mathrm{ad}(A)^2)$$

on $\mathfrak{sl}(2, \mathbb{R})$ equals -8 times the quadratic form defined by (15).) One reason for this choice of inner product is that if A denotes one of the the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

respectively, then $A \cdot A$ equals 1, 1, and -1 respectively.

Any inner product on a tangent space to a Lie group extends uniquely to a left-invariant pseudo-Riemannian metric. An inner product on the tangent space to the identity element e which is invariant under the *adjoint representation* Ad extends to a unique pseudo-Riemannian metric invariant under *both* left- and right-multiplications. In this way the inner product corresponding to (15) defines a bi-invariant Lorentzian metric on $\mathrm{PSL}(2, \mathbb{R})$, which has constant curvature -1 . This Lorentzian manifold is *anti-de Sitter space* and denoted AdS^3 .

Under the identification (5), the *Lorentzian cross-product*

$$\begin{aligned}\mathbb{R}^{2,1} \times \mathbb{R}^{2,1} &\longrightarrow \mathbb{R}^{2,1} \\ (A, B) &\longmapsto A \times B\end{aligned}$$

corresponds to $1/2$ the Lie bracket in $\mathfrak{sl}(2, \mathbb{R})$. The *scalar triple product* in $\mathbb{R}^{2,1}$ identifies with the 3×3 determinant:

$$(A \times B) \cdot C = \text{Det}(A, B, C)$$

and satisfies, for example, the vector identity

$$(A \times B) \cdot C = (B \times C) \cdot A.$$

In particular if $A, B \in \mathfrak{sl}(2, \mathbb{R})$, then $A \times B = \frac{1}{2}[A, B]$ is orthogonal to both A and B :

$$(7) \quad (A \times B) \cdot A = (A \times B) \cdot B = 0.$$

2.2.2. Cartan decomposition. The Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of AdS^3 is easy to describe in the $\text{PSL}(2, \mathbb{R})$ -model. Namely, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$, and (3) implies the Cartan involution is the permutation switching the two summands. The isotropy algebra \mathfrak{h} is the 1-eigenspace of the Cartan involution and is the diagonally embedded subalgebra

$$\begin{aligned}\mathfrak{sl}(2, \mathbb{R}) &\hookrightarrow \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \\ A &\longmapsto (A, A)\end{aligned}$$

tangent to the isotropy group $H = \text{Inn}(\text{PSL}(2, \mathbb{R}))$. Its orthogonal complement \mathfrak{m} is the (-1) -eigenspace of the Cartan involution and consists of the image of the embedding

$$\begin{aligned}\mathfrak{sl}(2, \mathbb{R}) &\hookrightarrow \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \\ A &\longmapsto (A, -A)\end{aligned}$$

and models the tangent space

$$T_e(\text{AdS}^3) \cong \mathbb{R}^{2,1} \cong \mathfrak{sl}(2, \mathbb{R}).$$

2.2.3. Geodesics. AdS^3 is a *Lorentzian symmetric space* and we recall several general facts about symmetric spaces.

Choose the identity element $e \in \text{PSL}(2, \mathbb{R})$ as basepoint in AdS^3 and identify its tangent space $T_e(\text{PSL}(2, \mathbb{R}))$ with the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ as above.

Geodesics through e in AdS^3 correspond to one-parameter subgroups in $\text{PSL}(2, \mathbb{R})$. In particular the Lie-theoretic exponential map

$$\mathfrak{sl}(2, \mathbb{R}) \xrightarrow{\text{exp}} \text{PSL}(2, \mathbb{R})$$

agrees with the Levi-Civita exponential map:

$$T_e(\text{AdS}^3) \xrightarrow{\text{Exp}_e} \text{AdS}^3$$

under the identification of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with the tangent space:

$$\mathfrak{sl}(2, \mathbb{R}) \xrightarrow{\cong} T_e(\text{PSL}(2, \mathbb{R})) \xrightarrow{\cong} T_e(\text{AdS}^3).$$

(Compare [15], pp. 224–226, and [10]. In particular [15], §V.6 and Exercise 3 (p. 227) describe symmetric spaces arising from Lie groups with the action of left- and right-multiplications.)

2.2.4. *Transvections.* Recall in a symmetric space G/H , the *transvection* along a geodesic $\mathbb{R} \xrightarrow{\gamma} G/H$ is the one-parameter group of isometries $\Phi_\gamma(t) \in G$ such that:

- $\Phi_\gamma(t) : \gamma(s) \mapsto \gamma(s + t)$;
- Parallel transport along γ is the differential

$$(D\Phi_\gamma(t))_{\gamma(s)} : T_{\gamma(s)}(G/H) \longrightarrow T_{\gamma(s+t)}(G/H).$$

As in §2.2.2, the isotropy algebra \mathfrak{h} acting on AdS^3 is the diagonal

$$\mathfrak{h} = \{(\xi, \xi) \mid \xi \in \mathfrak{sl}(2, \mathbb{R})\}$$

and its orthogonal complement \mathfrak{m} equals

$$\mathfrak{m} = \{(\xi, -\xi) \mid \xi \in \mathfrak{sl}(2, \mathbb{R})\}.$$

Therefore the transvection along the geodesic $\gamma(t)$ corresponding to a one-parameter subgroup $\exp(t\xi) \in G$ equals:

$$(8) \quad \text{AdS}^3 \xrightarrow{\Phi_\gamma(t)} \text{AdS}^3$$

$$x \mapsto \exp\left(\frac{t}{2}\xi\right) x \exp\left(\frac{t}{2}\xi\right)$$

2.3. **Totally geodesic subspaces and Lie triples.** Identify $\mathfrak{m} \cong T_e(\text{AdS}^3)$ with $\mathfrak{sl}(2, \mathbb{R})$ to realize subspaces of the tangent space as subspaces of $\mathfrak{sl}(2, \mathbb{R})$.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition. Then totally geodesic subspaces correspond to *Lie triples*, that is, subspaces $\mathfrak{s} \subset \mathfrak{m}$ such that

$$[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \subset \mathfrak{s}.$$

(Compare [10] and [15], §IV.7.)

In AdS^3 , any 2-plane \mathfrak{s} in \mathfrak{m} is a Lie triple. This is easy to see as follows. First of all, since $\dim(\mathfrak{s}) = 2$ and $[\cdot, \cdot]$ is skew-symmetric and bilinear, $\dim([\mathfrak{s}, \mathfrak{s}]) \leq 1$. (In fact $\dim([\mathfrak{s}, \mathfrak{s}]) = 1$.)

There are three types of 2-planes $\mathfrak{s} \subset \mathfrak{m} \cong \mathfrak{sl}(2, \mathbb{R})$, depending on the restriction of the inner product. Null 2-planes $\mathfrak{s} \subset \mathfrak{sl}(2, \mathbb{R})$

are Borel subalgebras, each of which is conjugate to the algebra \mathfrak{b} of upper-triangular matrices. Then $[\mathfrak{s}, \mathfrak{s}] \cong [\mathfrak{b}, \mathfrak{b}]$ is a line spanned by a nonzero element (corresponding to a nilpotent upper-triangular matrix in $\mathfrak{sl}(2, \mathbb{R})$). Furthermore

$$[[\mathfrak{b}, \mathfrak{b}], \mathfrak{b}] = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$$

as desired.

The remaining two cases follow from the vector identity (7). When \mathfrak{s} is spacelike, a normal vector is elliptic, and a normal vector v of unit length will satisfy the matrix equation $v^2 = -\mathbf{1}$. Furthermore $\text{ad}(v)$ acts on $\mathcal{S} = v^\perp$ by rotation of order 4. When \mathcal{S} is timelike, a unit length normal vector satisfies $v^2 = \mathbf{1}$ and $\text{ad}(v)$ acts on \mathcal{S} by a linear map with eigenvalues ± 1 .

Totally geodesic surfaces through the basepoint $e \in \text{AdS}^3$ are precisely the images $\exp(\mathfrak{s}) = \text{Exp}_e(\mathfrak{s})$, where we identify $T_e(\text{AdS}^3)$ with \mathfrak{m} . Thus, for any point $g \in \text{AdS}^3$ and every 2-plane $\mathfrak{s} \subset T_g(\text{AdS}^3)$,

$$S = \text{Exp}_g(\mathfrak{s})$$

is the unique totally geodesic surface $S \ni g$ with $T_g(S) = \mathfrak{s}$. Corresponding to the three types of planes in $\mathfrak{sl}(2, \mathbb{R})$, there are three types of totally geodesic surfaces of AdS^3 . (Compare §2.3.1, §2.3.3, §3.5, §2.2, and §3.5.2.)

2.3.1. Hyperbolic planes in AdS^3 . A point $g \in \text{AdS}^3$ determines a *dual* spacelike totally geodesic surface $g^* \subset \text{AdS}^3$ which is spacelike, as follows.

At the origin $e \in \text{AdS}^3$, timelike geodesics correspond to elliptic one-parameter subgroups, all of the form

$$\{hE_\theta h^{-1} \mid \theta \in \mathbb{R}\}$$

where

$$(9) \quad E_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

An elliptic one-parameter subgroup is *periodic* in $\text{SL}(2, \mathbb{R})$ with period 2π and periodic in $\text{PSL}(2, \mathbb{R}) \approx \text{AdS}^3$ with period π .

For an arbitrary point $g \in \text{AdS}^3$, the set of timelike geodesics from g correspond to cosets $ghE_\theta h^{-1}$ of elliptic one-parameter subgroups, where the parametrization by θ is chosen so that $E_\theta = e$ if and only if $\theta \in \pi\mathbb{Z}$. In particular each timelike geodesic in AdS^3 is closed and has length π .

Denote the set of *involutions* (elements of order two) in $\mathrm{PSL}(2, \mathbb{R})$ by Inv . The inverse image $\widehat{\mathrm{Inv}}$ of Inv in $\mathrm{SL}(2, \mathbb{R})$ consists of elements of order four, which have the form $hE_\theta h^{-1}$ where

$$\theta \equiv \pm\pi/2 \pmod{2\pi}.$$

The *hyperbolic plane dual to g* is the isometrically embedded \mathbf{H}^2 defined as:

$$\begin{aligned} g^* &:= g \mathrm{Inv} = \mathrm{Inv} g \\ &= \mathrm{Exp}_g \left(\{ \mathbf{v} \in T_g(\mathrm{AdS}^3) \mid \mathbf{v} \cdot \mathbf{v} = -\pi^2/4 \} \right). \end{aligned}$$

Here is an elegant description of this duality. Let $g \in \mathrm{AdS}^3$. Then the fixed set $\mathrm{Fix}(\iota_g)$ of the symmetry ι_g in g equals the disjoint union $\{g\} \cup g^*$.

Explicitly, $e^* = \mathrm{Inv}$ is the image of the isometric embedding

$$(10) \quad \begin{aligned} \mathbf{H}^2 &\hookrightarrow \mathrm{PSL}(2, \mathbb{R}) \approx \mathrm{AdS}^3 \\ x + iy &\longmapsto \pm \frac{1}{y} \begin{bmatrix} x & -(x^2 + y^2) \\ 1 & -x \end{bmatrix} \end{aligned}$$

of the upper halfplane $y > 0$. The positive imaginary axis $x = 0 < y$ defines a geodesic $\mathbf{H}^1 \subset \mathbf{H}^2$ where

$$(11) \quad e^t i \longmapsto \mathbf{J}(t) := \begin{bmatrix} 0 & -e^t \\ e^{-t} & 0 \end{bmatrix}.$$

2.3.2. Indefinite planes in AdS^3 . Totally geodesic indefinite 2-planes are isometrically embedded copies of AdS^2 , which are homeomorphic to Möbius bands, and therefore nonorientable.

One can see this as follows. The isometry group G^0 acts transitively on pairs (g, \mathfrak{s}) , where $g \in \mathrm{AdS}^3$ is a point and $\mathfrak{S} \subset T_p(\mathrm{AdS}^3)$ is an indefinite 2-plane. Thus we may assume that $g = e$ and \mathfrak{s} is the subspace of $T_e(\mathrm{AdS}^3) \cong \mathfrak{sl}(2, \mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}.$$

In $\mathrm{SL}(2, \mathbb{R})$ this subspace exponentiates to the subset \mathfrak{S} of matrices having equal diagonal entries with determinant 1:

$$(12) \quad \mathfrak{S} := \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \mid a^2 - bc = 1 \right\}$$

which is a one-sheeted hyperboloid. Its image in AdS^3 is the quotient of this hyperboloid by the antipodal map

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix},$$

with quotient homeomorphic to a Möbius band. Since it is nonorientable, and AdS^3 is orientable, it is one-sided in AdS^3 .

2.3.3. *Null planes in AdS^3* . Tangent to degenerate 2-planes in $T_g(\text{AdS}^3)$ are degenerate totally geodesic subspaces, which we call *null planes*. A convenient model is the *Borel subgroup* $\mathfrak{B} = \exp(\mathfrak{b})$ consisting of upper-triangular matrices. The geodesics through e in this null plane are the one-parameter subgroups:

$$(13) \quad \exp\left(t \begin{bmatrix} \alpha & \beta \\ 0 & -\alpha \end{bmatrix}\right) = \begin{bmatrix} e^{t\alpha} & \frac{\beta}{\alpha} \sinh(t\alpha) \\ 0 & e^{-t\alpha} \end{bmatrix}$$

The commutator subgroup consists of unipotent upper-triangular matrices, the cosets of which are null geodesics. They define a foliation tangent to the line field detecting the degeneracy of the induced Lorentzian structure. The complement of one of these null geodesics is the disjoint union of two *null halfplanes*. Just as in Minkowski space, the ambient orientation of AdS^3 determines a way to consistently choose a null halfplane bounded by a null geodesic ℓ . (See §2.3.4.)

A null geodesic ℓ in AdS^3 lies in a unique null plane $P(\ell)$. Given any point $g \in \ell$, the null 2-plane

$$T_g(\ell)^\perp \subset T_g(\text{AdS}^3)$$

is the unique null plane in the tangent space containing the line $T_p(\ell)$. The totally geodesic subspace

$$P(\ell, g) := \text{Exp}_g(T_g(\ell)^\perp)$$

is the desired null plane. To see that it is independent of g , choose coordinates so that P consists of the image of upper-triangular matrices in $\text{PSL}(2, \mathbb{R})$, and let $\Phi_\ell(t)$ denote the transvection along ℓ , taking $g \in \ell$ to $h \in \ell$. Using the formula (8) for transvections,

$$\begin{bmatrix} 1 & t/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & t/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b + \left(\frac{a+a^{-1}}{2}\right)t \\ 0 & a^{-1} \end{bmatrix} \in P(g, \ell)$$

so that

$$P(h, \ell) = \Phi_\ell(t)P(g, \ell) = P(g, \ell)$$

as desired. Henceforth denote $P(g, \ell)$ by just $P(\ell)$.

2.3.4. *Consistent choice of orientations.* The complement $P(\ell) \setminus \ell$ decomposes as the disjoint union of two null halfplanes; here is how to choose one of them, which we call the *wing* $\mathcal{W}(\ell)$.

First assume an orientation on AdS^3 , which is given by a volume form Det on the tangent space $T_g(\text{AdS}^3)$. Choose a future-pointing vector \mathbf{v} tangent to ℓ , and another future-pointing vector \mathbf{u} in the complement of $P(\ell)$. Then, for $p \in \ell$, the *wing*

$$(14) \quad \mathcal{W}(\ell) := \text{Exp}_g\left(\{\mathbf{w} \in T_g P(\ell) \mid \text{Det}(\mathbf{v}, \mathbf{u}, \mathbf{w}) > 0\}\right)$$

is a null halfplane bounded by ℓ . Compare [3], §3.2.

2.4. **AdS-crooked planes.** Now we define crooked planes in anti-de Sitter space, following Danciger-Guéritaud-Kassel [6]. We begin by giving several definitions, the first one of which is analogous to the original definition in Minkowski space. From that we give alternate definitions, in terms of the vertex/spine pair, and a dual definition which doesn't have a direct analogue in Minkowski space. Then we describe how an AdS-crooked plane \mathbf{C} determines a crooked plane \mathcal{C} in Minkowski space, by a *tangent cone construction*: \mathcal{C} is the tangent cone of \mathbf{C} at its vertex p , where the tangent space $T_p(\text{AdS}^3)$ is identified with Minkowski space. From that, it is immediate that $\mathbf{C} = \text{Exp}_p(\mathcal{C})$, establishing (1) of the Main Theorem.

The main difference between crooked planes in AdS^3 and crooked planes in Minkowski space is that the *particles*, the timelike geodesics lying on the stem are compact (homeomorphic to circles) in AdS^3 but, in Minkowski space, are lines.

2.4.1. *Stems, hinges, wings and the spine.* Following [6], given a point g and a unit spacelike vector $\mathbf{s} \in T_g(\text{AdS}^3)$, we define an *AdS-crooked plane*, as follows. The orthogonal complement $\mathbf{s}^\perp \subset T_g(\text{AdS}^3)$ is a timelike 2-plane, and it exponentiates to a totally geodesic indefinite plane

$$S := \text{Exp}_g(\mathbf{s}^\perp) \ni g.$$

The *stem* $\text{Stem}(p, \mathbf{s})$ is defined as the union of the non-spacelike geodesics in S . We call the timelike geodesics lying in S the *particles*.

$\text{Stem}(p, \mathbf{s})$ is bounded by two null geodesics h_1, h_2 , the *hinges*. Each hinge lies in a unique null plane, as in §2.3.3. The stem is the union of the two hinges and the particles.

The *wings* $W_i := \mathcal{W}(h_i)$ are defined as the null halfplanes associated to the hinges, as in (14). The *crooked plane* $\mathbf{C}(g, \mathbf{s})$ is then defined as the union:

$$\mathbf{C}(g, \mathbf{s}) := \text{Stem}(g, \mathbf{s}) \cup \mathcal{W}_1 \cup \mathcal{W}_2.$$

The *spine* is the unique spacelike geodesic contained in the crooked plane, and equals the image σ of:

$$\begin{aligned}\mathbb{R} &\longrightarrow \text{AdS}^3 \\ t &\longmapsto \text{Exp}_g(ts)\end{aligned}$$

The spine lies on the union of two wings $\mathcal{W}_1 \cup \mathcal{W}_2$, bisecting each one. The vertex g and the spine σ completely determine the crooked plane.

Since the identity component of the isometry group G^0 acts transitively on pairs (g, \mathbf{s}) , all AdS -crooked planes are equivalent under the orientation-preserving isometry group of AdS^3 .

2.4.2. The dual definition. Using the duality between points in AdS^3 and hyperbolic planes, here is an alternate definition of crooked planes in AdS^3 . Dual to the vertex g is a hyperbolic plane $g^* \subset \text{AdS}^3$. The stem $\text{Stem}(g, \mathbf{s})$ intersects g^* in a geodesic $\ell \subset g^*$. The pair (ℓ, g^*) , consisting of a hyperbolic plane g^* and a geodesic $\ell \subset g^*$ also suffices to determine the crooked plane.

To recover the vertex-spine pair from a pair (H, ℓ) , where H is a hyperbolic plane containing a geodesic ℓ , take the vertex to be dual to H . A unique spacelike plane S contains ℓ and is orthogonal to H . Since S is orthogonal to H , it contains g . Indeed, S is characterized as the unique spacelike plane containing $\ell \cup \{g\}$. The intersection $H \cap S$ is a spacelike geodesic in S , which is the spine σ of the crooked plane. Thus the data $(g, \pm \mathbf{s})$ is equivalent to the data (H, ℓ) . This dual description is absent from the classic construction of crooked planes in Minkowski space.

2.4.3. The tangent cone crooked plane. An AdS -crooked plane with vertex g is *star-shaped* about g : that is, it is the union of geodesic rays emanating from g . By definition, its *tangent cone* $\mathcal{C} \subset T_g(\text{AdS}^3)$ at g consists of all vectors $\mathbf{v} \in T_g(\text{AdS}^3)$ tangent to smooth rays emanating from g . (It is a *cone* in that it is invariant under positive scalar multiplications in the tangent space.) In particular,

$$\mathcal{C} = \text{Exp}_g(\mathcal{C})$$

Moreover, under the identification of the tangent space $T_g(\text{AdS}^3)$ with Minkowski space, $\mathcal{C} \subset T_g(\text{AdS}^3)$ identifies with a crooked plane with vertex $0 \in T_g(\text{AdS}^3)$.

2.5. The double covering space $\widehat{\text{AdS}}$. The (unique) double covering space, denoted $\widehat{\text{AdS}}$, corresponds to $\text{SL}(2, \mathbb{R})$, consisting of real 2×2

matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $ad - bc = 1$. Since $\widehat{\text{AdS}} = \text{SL}(2, \mathbb{R})$ is a group, it has a natural basepoint, the identity matrix $\mathbf{1} \in \text{SL}(2, \mathbb{R})$. The deck transformation is the involution given by multiplication by $-\mathbf{1}$, which we denote by

$$\hat{g} \mapsto -\hat{g}.$$

The action of G^0 lifts to the action of the group $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ by left- and right-multiplication (1), whose kernel is the diagonally embedded central subgroup of order two generated by $(-\mathbf{1}, -\mathbf{1})$.

In $\widehat{\text{AdS}}$, the timelike geodesics are all closed and have length 2π .

Unlike AdS^3 , not every pair of points in $\widehat{\text{AdS}}$ is connected by a geodesic. Let $\mathbf{1} \in \widehat{\text{AdS}}$ be the identity matrix. Then the point corresponding to $g \in \text{SL}(2, \mathbb{R})$ is connected by a geodesic to $\mathbf{1}$ if and only if $\text{tr}(g) > -2$ or g corresponds to $-\mathbf{1}$.

The duality between points and hyperbolic planes is somewhat different as well. Choose a point $\hat{g} \in \widehat{\text{AdS}}$ covering $g \in \text{AdS}^3$ and a lift $\hat{g}^* \subset \widehat{\text{AdS}}$ of the dual hyperbolic plane $g^* \subset \text{AdS}$. The preimage in $\widehat{\text{AdS}}$ of g^* is the disjoint union $\hat{g}^* \cup (-\hat{g}^*)$, where \hat{g}^* is an (arbitrarily chosen) component.

Then the symmetry $\iota_{\hat{g}}$ in \hat{g} fixes both \hat{g} and $-\hat{g}$ but interchanges the planes \hat{g}^* and $-\hat{g}^*$. On the other hand,

$$\begin{aligned} \widehat{\text{AdS}} &\longrightarrow \widehat{\text{AdS}} \\ \hat{h} &\longmapsto -\iota_{\hat{g}}(\hat{h}) \end{aligned}$$

is another reflection which fixes each of the two planes $\pm\hat{g}^*$ obtained by lifting g^* , but interchanges the *points* \hat{g} and $-\hat{g}$.

2.5.1. *Lifting crooked planes.* The inverse image $\widehat{\mathcal{C}}(\hat{g}, \mathbf{s})$ of $\mathcal{C}(g, \mathbf{s})$ under the double covering $\widehat{\text{AdS}} \longrightarrow \text{AdS}^3$ is connected. Choose one lift of g to call the *vertex* \hat{g} of $\widehat{\mathcal{C}}(\hat{g}, \mathbf{s})$ and call the other one the *covertex* $\hat{g}' = -\hat{g}$. The inverse image of each hinge has two components, one containing the vertex and the other containing the covertex. Call the component containing the vertex a *hinge* and the component containing the covertex a *cohinge*. The inverse image of the stem contains closed timelike geodesics all of which contain the vertex and the covertex. Each of the two wings of $\mathcal{C}(\hat{g}, \mathbf{s})$ has two preimages as well, determined by which hinge or cohinge bounds them. We call the lifts of the wings which pass through the vertex *wings* of the $\widehat{\text{AdS}}$ -crooked plane, and the lifts of the wings through the covertex the *cowings*. Similarly the spine

lifts to two spacelike geodesics; the one containing the vertex we call the *spine* of the $\widehat{\text{AdS}}$ crooked plane and the one containing the covertex its *cospine*.

Although AdS -crooked planes are star-shaped about their vertices, their lifts are not. If \mathcal{C} is the tangent cone of an $\widehat{\text{AdS}}$ crooked plane $\widehat{\mathcal{C}}$ at a vertex \hat{g} , then $\widehat{\text{Exp}}_{\hat{g}}(\mathcal{C})$ contains neither cohingens nor cowings. However,

$$\begin{aligned}\widehat{\mathcal{C}} &= \widehat{\text{Exp}}_{\hat{g}}(\mathcal{C}) \cup -\widehat{\text{Exp}}_{\hat{g}}(\mathcal{C}) \\ &= \widehat{\text{Exp}}_{\hat{g}}(\mathcal{C}) \cup \widehat{\text{Exp}}_{-\hat{g}}(-\mathcal{C})\end{aligned}$$

where $-\mathcal{C}$ denotes the tangent cone of $\widehat{\text{AdS}}$ at the covertex $-\hat{g}$.

3. EMBEDDING IN THE EINSTEIN UNIVERSE

The double covering space $\widehat{\text{AdS}}$ of AdS^3 embeds in Ein^3 . This is analogous to the embedding of a double covering space of \mathbb{H}^2 in S^2 as the upper and lower hemispheres. Reflection in the equator permutes these hemispheres, acting as the deck involution in this covering space. The equator is the fixed set of the reflection, and models conformal Riemannian geometry in one dimension lower. Similarly conformal inversion $I_{\mathcal{S}}$ in the unit (Lorentzian) sphere is the deck involution in $\widehat{\text{AdS}}$ for the double covering $\widehat{\text{AdS}} \rightarrow \text{AdS}^3$. The complement of $\widehat{\text{AdS}}$ in Ein^3 is exactly the set $\text{Fix}(I_{\mathcal{S}})$ of fixed points, which is just $\text{Ein}^2 \subset \text{Ein}^3$, analogous to the Riemannian case.

We briefly review the geometry of the Einstein universe, following [2]. See also [11, 12, 16, 19].

3.1. Embeddings in Ein^3 . Following [2], *Einstein n -space* Ein^n is the projectivized nullcone $\mathbb{P}(\mathfrak{N}) \subset \mathbb{P}(\mathbb{R}^{n,2})$ in the real inner product space $\mathbb{R}^{n,2}$. We follow the notation from [2] for $n = 3$: points in $\mathbb{R}^{3,2}$ have coordinates (X, Y, Z, U, V) and the quadratic form is:

$$(15) \quad Q(X, Y, Z, U, V) := X^2 + Y^2 - Z^2 - UV.$$

Thus the nullcone in $\mathbb{R}^{3,2}$ equals:

$$\mathfrak{N} := \left\{ (X, Y, Z, U, V) \in \mathbb{R}^{3,2} \mid Q(X, Y, Z, U, V) = 0 \right\}$$

and Ein^3 consists of all 1-dimensional linear subspaces of $\mathbb{R}^{3,2}$ contained in \mathfrak{N} . Isotropic 2-planes in $\mathbb{R}^{3,2}$ projectivize to *photons*, which are embedded smooth circles in Ein^3 . The union of all photons through a point $p \in \text{Ein}^3$ is the *lightcone* $L(p) \subset \text{Ein}^3$. The complement

$$\text{Min}(p) := \text{Ein}^3 \setminus L(p)$$

is the *Minkowski patch* associated to p .

3.2. Minkowski space. Define, respectively, the *improper point* and the *origin* :

$$\begin{aligned} p_\infty &:= [0 : 0 : 0 : 1 : 0] \in \mathbf{Ein}^3 \\ p_0 &:= [0 : 0 : 0 : 0 : 1] \in \mathbf{Ein}^3. \end{aligned}$$

The Minkowski patch $\mathbf{Min}(p_\infty)$ is the image of the embedding of *three-dimensional Minkowski space* in \mathbf{Ein}^3 (see [2]):

$$(16) \quad \begin{aligned} \mathbf{E}^3 &\hookrightarrow \mathbf{P}(\mathfrak{N}) = \mathbf{Ein}^3 \\ (x, y, z) &\mapsto [x : y : z : x^2 + y^2 - z^2 : 1] \end{aligned}$$

In homogeneous coordinates, this Minkowski patch $\mathbf{Min}(p_\infty)$ is defined by $V \neq 0$, and the point p_0 corresponds to the *origin* $(0, 0, 0) \in \mathbf{E}^3$. The ordered pair (p_∞, p_0) makes \mathbf{E}^3 into a *Lorentzian vector space*. The *affine lightcone* (the “usual” light cone in Minkowski space) $L^{\text{aff}}(p_0)$ is the intersection $\mathbf{E}^3 \cap L(p_0) = L(p_0) \setminus L(p_\infty)$ defined as

$$\{(x, y, z) \in \mathbf{E}^3 \mid x^2 + y^2 - z^2 = 0\}.$$

The (Lorentzian) *unit sphere* $\mathcal{S} \subset \mathbf{E}^3$ is the subset

$$\mathcal{S} := \{(x, y, z) \in \mathbf{E}^3 \mid x^2 + y^2 - z^2 = 1\}.$$

Analogous to inversion in Euclidean geometry, *inversion in \mathcal{S}* :

$$\begin{aligned} \mathbf{E}^3 \setminus L(p_0) &\xrightarrow{I_{\mathcal{S}}} \mathbf{E}^3 \\ (x, y, z) &\longmapsto \frac{1}{x^2 + y^2 - z^2}(x, y, z) \end{aligned}$$

is conformal, and we use this inversion to define a coordinate atlas on \mathbf{Ein}^3 . It is the restriction of the involution of \mathbf{Ein}^3 defined by:

$$(17) \quad \begin{aligned} \mathbf{Ein}^3 &\xrightarrow{I_{\mathcal{S}}} \mathbf{Ein}^3 \\ [X : Y : Z : U : V] &\longmapsto [X : Y : Z : V : U] \end{aligned}$$

interchanging p_0 and p_∞ . The image

$$\mathfrak{I} := I_{\mathcal{S}}(L^{\text{aff}}(p_0) \setminus \{p_0\}) = L(p_\infty) \cap \mathbf{Min}(p_0) \setminus \{p_\infty\}$$

is called the *inverse cone*. The *ideal circle* is the intersection

$$(18) \quad C_\infty := L(p_0) \cap L(p_\infty),$$

consisting of ideal points of photons through the origin p_0 . It (together with the improper point p_∞) compactifies both the unit-sphere \mathcal{S} and the lightcone $L^{\text{aff}}(p_0)$. Alternatively, C_∞ consists of ideal points of photons through the improper point p_∞ .

3.2.1. *Topology of Ein^3 .* Topologically, Ein^3 is a mapping torus of the antipodal map of S^2 , decomposing into four strata:

- (Dimension 3) The Minkowski patch \mathbf{E}^3 is a 3-cell;
- (Dimension 2) The inverse cone \mathfrak{J} consists of two disjoint annuli;
- (Dimension 1) The ideal circle C_∞ is a 1-sphere;
- (Dimension 0) The improper point p_∞ is a 0-cell.

For more details see [3, 16].

Because of (18), the union $\text{Min}(p_0) \cup \text{Min}(p_\infty)$ equals the complement $\text{Ein}^3 \setminus C_\infty$. Since Ein^3 is not a sphere, one needs at least three Minkowski patches to cover it. If $p \in \text{Ein}^3$, then

$$\text{Min}(p_0) \cup \text{Min}(p_\infty) \cup \text{Min}(p) = \text{Ein}^3$$

if and only if p lies on a timelike line through p_0 . This is the condition that the *Maslov index* of the triple (p_0, p_∞, p) equals ± 1 .

The involution $I_{\mathcal{G}}$ extends to Ein^3 by mapping p_0 to p_∞ , mapping $L^{\text{aff}}(p_0) \setminus \{p_0\}$ to \mathfrak{J} , and fixing C_∞ pointwise. Denote this involution of Ein^3 by $I_{\mathcal{G}}$ as well, when there is no danger of confusion.

3.3. **Embedding $\widehat{\text{AdS}}$.** The mapping

$$(19) \quad \begin{aligned} \text{SL}(2, \mathbb{R}) &\xrightarrow{\Psi} \text{P}(\mathfrak{N}) \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\longmapsto [a - d : b + c : b - c : a + d - 2 : a + d + 2] \end{aligned}$$

embeds $\text{SL}(2, \mathbb{R})$ in Ein^3 . The identity matrix $\mathbf{1} \in \text{SL}(2, \mathbb{R})$ maps to the point p_0 corresponding to the origin $(0, 0, 0)$ in Minkowski space \mathbf{E}^3 , under (16). The other central element $-\mathbf{1} \in \text{SL}(2, \mathbb{R})$ maps to p_∞ .

3.3.1. *Equivariance under the deck involution.* The double covering $\widehat{\text{AdS}} \rightarrow \text{AdS}^3$ corresponds to the double covering $\text{SL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$. In this model, the deck involution is multiplication by $-\mathbf{1}$ in $\text{SL}(2, \mathbb{R})$, and Ψ relates this involution to the inversion $I_{\mathcal{G}}$ defined by (17). To see this, write:

$$\begin{aligned} M &:= \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \\ \Psi(M) &= [X : Y : Z : U : V] \\ &= [a - d : b + c : b - c : a + d - 2 : a + d + 2], \end{aligned}$$

so (19) implies:

$$\begin{aligned}\Psi(-M) &= [-X : -Y : -Z : -(a+d) - 2 : -(a+d) + 2] \\ &= [X : Y : Z : (a+d) + 2 : (a+d) - 2] \\ &= [X : Y : Z : V : U].\end{aligned}$$

3.3.2. *The boundary of $\widehat{\text{AdS}}$.* The double cover $\widehat{\text{AdS}}$ of AdS^3 is bounded by Ein^2 , which is a 1-sided surface in Ein^3 . This surface consists of all points in Ein^3 fixed under the involution $I_{\mathcal{S}}$ and the quotient mapping

$$\text{Ein}^3 \longrightarrow \text{Ein}^3/I_{\mathcal{S}} = \text{AdS}^3 \cup \text{Ein}^2$$

is a double covering branched over Ein^2 . Using the usual compactification of AdS^3 as a closed solid torus (the 3-manifold-with-boundary $D^2 \times S^1$), Ein^3 is homeomorphic to the identification space of $D^2 \times S^1$ by an orientation-reversing involution indentifying the boundary to itself.

The embedding of $\widehat{\text{AdS}} \xrightarrow{\Psi} \text{Ein}^3$ and the involution $I_{\mathcal{S}}$ intimately relate. The complement of the image of Ψ is an Einstein hypersphere Ein^2 , consisting of fixed points of the involution $I_{\mathcal{S}}$. Clearly points in the image of Ψ are not fixed by $I_{\mathcal{S}}$; thus their homogeneous coordinates satisfy $U \neq V$. Conversely,

$$\begin{aligned}a &= \frac{2X + U + V}{V - U} \\ b &= \frac{2(Y + Z)}{V - U} \\ c &= \frac{2(Y - Z)}{V - U} \\ d &= \frac{-2X + U + V}{V - U}\end{aligned}$$

defines an element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{R})$$

mapping to $[X : Y : Z : U : V]$ if $U \neq V$.

The hyperplane in $\mathbb{R}^{3,2}$ defined by $U = V$ has signature $(2, 2)$ and therefore its image is a 2-dimensional Einstein subspace, which we henceforth call Ein^2 . It equals the union

$$\mathcal{S} \cup C_{\infty} = \text{Fix}(I_{\mathcal{S}}).$$

Although the unit sphere \mathcal{S} (that is, the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$) disconnects \mathbb{E}^3 , its closure Ein^2 does *not* disconnect Ein^3 .

3.3.3. *Ideal points for the embedding of $\widehat{\text{AdS}} \hookrightarrow \text{Ein}^3$.* The lightcone

$$L(p_\infty) = I_{\mathcal{S}}(L(p_0))$$

is the complement of Minkowski space \mathbf{E}^3 in Ein^3 . It is the union of all null geodesics through the improper point $p_\infty = I_{\mathcal{S}}(p_0)$. Since p_∞ corresponds to the matrix $-\mathbf{1} \in \text{SL}(2, \mathbb{R})$, the lightcone $L(p_\infty)$ corresponds to the cosets of unipotent one-parameter subgroups through $-\mathbf{1}$. These are precisely the elements of $\text{SL}(2, \mathbb{R})$ having trace -2 .

As in [2], this lightcone meets $L(p_0)$ in the ideal circle C_∞ . Thus the boundary of $\widehat{\text{AdS}}$ is the Lorentzian unit sphere $\mathcal{S} \subset \mathbf{E}^3$ compactified with the ideal circle C_∞ . The union $\mathcal{S} \cup C_\infty$ is Ein^2 , which is topologically a torus, which is a 1-sided surface in the nonorientable 3-manifold Ein^3 .

3.4. **Einstein hyperspheres and the boundary of AdS^3 .** Because $\text{PSL}(2, \mathbb{R})$ is homeomorphic to an open solid torus $D^2 \times S^1$, its boundary is a 2-torus $S^1 \times S^1$. Here we describe how this boundary relates to the conformal embedding of $\widehat{\text{AdS}}$ and the rulings of the quadric Ein^2 .

Since this theory seems not to be so well-known, we begin with a brief exposition of *singular projective transformations*. This is a simple case of the theory of *wonderful compactifications* developed by de Concini-Procesi [7].

3.4.1. *The boundary of $\text{PGL}(2, \mathbb{R})$.* A natural compactification of the group $\text{PGL}(2, \mathbb{R})$ is the projective space \mathbb{RP}^3 obtained by projectivizing the embedding

$$\text{GL}(2, \mathbb{R}) \hookrightarrow \text{Mat}_2(\mathbb{R}) \setminus \{\mathbf{0}\}$$

of the group of *invertible* 2×2 real matrices in the set of *nonzero* 2×2 real matrices. The induced map

$$\text{PGL}(2, \mathbb{R}) \hookrightarrow \mathbb{RP}^3$$

embeds $\text{PGL}(2, \mathbb{R})$ as an open dense subset of \mathbb{RP}^3 . The complement $\mathbb{RP}^3 \setminus \text{PGL}(2, \mathbb{R})$ consists of *singular projective transformations* of \mathbb{RP}^1 . It naturally identifies with $\mathbb{RP}^1 \times \mathbb{RP}^1$ and is homeomorphic to a torus.

A point in the the complement $\mathbb{RP}^3 \setminus \text{PGL}(2, \mathbb{R})$ is the projective equivalence class of a 2×2 real matrix M which is nonzero but singular. In other words, M is a 2×2 matrix of rank 1. The corresponding linear endomorphism of \mathbb{R}^2 has a 1-dimensional kernel and a 1-dimensional range, which may coincide. This pair of lines in \mathbb{R}^2 determines a point

$$(\text{Kernel}(M), \text{Image}(M)) \in \mathbb{RP}^1 \times \mathbb{RP}^1.$$

Two matrices giving the same pair are projectively equivalent. The corresponding *singular projective transformation* is only defined on $\mathbb{R}\mathbb{P}^1 \setminus \{\text{Kernel}(M)\}$ where it is the constant mapping

$$\mathbb{R}\mathbb{P}^1 \setminus \{\text{Kernel}(M)\} \longrightarrow \{\text{Image}(M)\}.$$

Here are two basic examples. The one-parameter semigroup of projective transformations defined by the diagonal matrix

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

converges, as $t \rightarrow +\infty$, to the singular projective transformation defined by the diagonal matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which corresponds to the ordered pair

$$([0 : 1], [1 : 0]) \in \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1.$$

Likewise the one-parameter semigroup of projective transformations

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

converges, as $t \rightarrow +\infty$, to the singular projective transformation defined by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which corresponds to the ordered pair

$$([1 : 0], [1 : 0]) \in \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1.$$

For more details see [13] and the references cited there.

The group $\text{PSL}(2, \mathbb{R})$ is one component of $\text{PGL}(2, \mathbb{R})$; the other component corresponds to matrices with negative determinant. Since the inclusion

$$\text{PSL}(2, \mathbb{R}) \hookrightarrow \overline{\text{PSL}(2, \mathbb{R})}$$

is a homotopy-equivalence, the compactification of $\text{PSL}(2, \mathbb{R})$ lifts to a compactification of the double covering

$$\text{SL}(2, \mathbb{R}) \longrightarrow \text{PSL}(2, \mathbb{R}).$$

Points on the boundary $\partial\text{SL}(2, \mathbb{R}) \approx S^1 \times S^1$ can be geometrically interpreted as follows. Choose an orientation $O_{\mathbb{R}^2}$ on \mathbb{R}^2 . The double covering $\widehat{\mathbb{R}\mathbb{P}^1}$ of $\mathbb{R}\mathbb{P}^1$ consists of *oriented lines*, that is, pairs (ℓ, O_ℓ) ,

where $\ell \in \mathbb{RP}^1$ is a line in \mathbb{R}^2 and O_ℓ is an orientation on ℓ . Then the boundary $\partial\mathrm{SL}(2, \mathbb{R})$ consists of ordered pairs of oriented lines

$$((\ell_1, O_1), (\ell_2, O_2)) \in \widehat{\mathbb{RP}^1} \times \widehat{\mathbb{RP}^1}$$

such that:

- $O_1 = O_2$ if $\ell_1 = \ell_2$;
- $O_{\mathbb{R}^2} = O_1 \oplus O_2$ with respect to the decomposition $\mathbb{R}^2 = \ell_1 \oplus \ell_2$ if $\ell_1 \neq \ell_2$.

3.4.2. *Rulings of Ein^2* . The fibers of the two projections

$$\partial\widehat{\mathrm{AdS}} \approx S^1 \times S^1 \longrightarrow S^1$$

correspond to the *rulings* of $\partial\widehat{\mathrm{AdS}} = \mathrm{Ein}^2$. (Compare Danciger [5].) In terms of the affine piece of Ein^2 , the Lorentzian unit-sphere \mathcal{S} defined by $x^2 + y^2 - z^2 = 1$, these are just the two families of null lines

$$(20) \quad \mathcal{S}_\theta^+ := \left\{ \begin{bmatrix} \cos(\theta) - \eta \sin(\theta) \\ \sin(\theta) + \eta \cos(\theta) \\ \eta \end{bmatrix} \mid \eta \in \mathbb{R} \right\}$$

and

$$\mathcal{S}_\theta^- := \left\{ \begin{bmatrix} \cos(\theta) + \eta \sin(\theta) \\ \sin(\theta) - \eta \cos(\theta) \\ \eta \end{bmatrix} \mid \eta \in \mathbb{R} \right\}$$

respectively, where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is fixed. These null lines have respective ideal points

$$\partial L_\theta^\pm := [0 : -\sin(\theta) : \cos(\theta) : \pm 1 : 0 : 0] \in C_\infty.$$

3.5. **$\widehat{\mathrm{AdS}}$ totally geodesic subspaces in Ein^3** . To relate AdS -crooked planes, crooked surfaces in Ein^3 and classic crooked planes in Minkowski space \mathbf{E}^3 , we first describe how $\widehat{\mathrm{AdS}}$ totally geodesic subspaces extend to submanifolds of Ein^3 .

3.5.1. *Geodesics*. As timelike geodesics in $\widehat{\mathrm{AdS}}$ correspond to elliptic one-parameter subgroups in $\mathrm{SL}(2, \mathbb{R})$, they are all closed and do not approach the boundary of $\widehat{\mathrm{AdS}}$ in Ein^3 .

Null geodesics in $\widehat{\mathrm{AdS}}$ have a single endpoint in $\partial\widehat{\mathrm{AdS}} = \mathrm{Ein}^2$. In our conformal Minkowski model of Einstein space, there are two types, depending on whether or not the null geodesic ν passes through the improper point. If $p_\infty \in \nu$, then ν lies on $\mathfrak{I} \cup \{p_\infty\}$; its single endpoint lies on the ideal circle C_∞ . If ν does not contain p_∞ , then it misses the ideal circle and contains a point of \mathfrak{I} . Its endpoint lies on \mathcal{S} .

The embedding Ψ defined in (19) maps an elliptic one-parameter subgroup to a timelike geodesic:

$$(21) \quad \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \xrightarrow{\Psi} [0 : 0 : \sin(\theta) : 1 - \cos(\theta) : 1 + \cos(\theta)]$$

which under (16) corresponds to the z -axis $x = y = 0$, with

$$z = \frac{\sin(\theta)}{1 + \cos(\theta)} = \tan(\theta/2),$$

mapping $-\pi < \theta < \pi$ to $-\infty < z < \infty$.

The embedding Ψ defined in (19) maps the (cosets of) unipotent one-parameter subgroups to photons:

$$(22) \quad \begin{aligned} & \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \xrightarrow{\Psi} H_1^+(t) := [0 : t : t : 0 : 4] \longleftrightarrow (0, t/4, t/4) \\ & - \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \xrightarrow{\Psi} H_1^-(t) := [0 : t : t : 4 : 0] \\ & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \xrightarrow{\Psi} H_2^+(t) := [0 : t : -t : 0 : 4] \longleftrightarrow (0, t/4, -t/4) \\ & - \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \xrightarrow{\Psi} H_2^-(t) := [0 : t : -t : 4 : 0] \end{aligned}$$

Their images h_1^+ , h_2^+ , correspond to null lines $x = y - z = 0$ and $x = y - z = 0$ respectively. Inverting in \mathcal{S} yield h_1^- , h_2^- respectively, which lie on the inverse cone.

Finally, Ψ maps cosets of hyperbolic one-parameter subgroups in $\mathrm{SL}(2, \mathbb{R}) = \widehat{\mathrm{AdS}}$ to spacelike arcs. An illustrative model is the intersection of the x -axis with the one-sheeted hyperboloid \mathcal{S} . The geodesic in $\widehat{\mathrm{AdS}}$ emanating from $\mathbf{1}$, which corresponds to the origin $(0, 0, 0) \in \mathbf{E}^3$, is just the line segment

$$(23) \quad \{-1 < x < 1\} \times \{(0, 0)\}$$

The geodesic in $\widehat{\mathrm{AdS}}$ emanating from $-\mathbf{1}$, which corresponds to the improper point p_∞ , divides into the union of two rays

$$(24) \quad \{x < -1\} \times \{(0, 0)\} \cup \{1 < x\} \times \{(0, 0)\},$$

separated by the improper point p_∞ at ∞ . These two rays correspond to the components of the complement of $-\mathbf{1}$ in the this coset. Explicitly:

$$(25) \quad \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \xrightarrow{\Psi} [\sinh(t) : 0 : 0 : \cosh(t) - 1 : \cosh(t) + 1]$$

which under (16) corresponds to the x -axis $y = z = 0$, with

$$x = \frac{\sinh(t)}{\cosh(t) + 1} = \tanh(t/2)$$

mapping $-\infty < t < \infty$ to $-1 < x < 1$. The other geodesic corresponds to its $I_{\mathcal{S}}$ -image:

$$(26) \quad - \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \xrightarrow{\Psi} [\sinh(t) : 0 : 0 : \cosh(t) + 1 : \cosh(t) - 1]$$

which under (16) corresponds to the x -axis $y = z = 0$, with

$$x = \frac{\sinh(t)}{\cosh(t) - 1} = \coth(t/2)$$

mapping $-\infty < t < 0 < \infty$ to $x < -1$ and $0 < t < \infty$ to $x > 1$.

These two $\widehat{\text{AdS}}$ geodesics, together with their two endpoints on \mathcal{S} constitute a *spacelike circle* in Ein^3 , as defined in [2].

3.5.2. *Totally geodesic surfaces.* Now we see how totally geodesic surfaces of $\widehat{\text{AdS}}$ embed in Ein^3 . Recall three types of totally geodesic surfaces in AdS^3 exist: definite planes, isometric to \mathbb{H}^2 ; degenerate planes, corresponding to cosets of Borel subgroups of $\text{SL}(2, \mathbb{R})$; indefinite planes, isometric to AdS^2 .

A definite plane is dual to a point $g \in \text{AdS}^3$ as discussed in §2.3.1. Here is a simple example. The plane e^* dual to the identity element $e \in \text{PSL}(2, \mathbb{R})$ is the set of involutions $\text{Inv} \subset \text{PSL}(2, \mathbb{R})$; see (10) for an explicit embedding of \mathbb{H}^2 in AdS^3 . In Ein^3 the two lifts of this hyperbolic plane are the two components of the hyperboloid $x^2 + y^2 - z^2 = -1$; their common boundary is the ideal circle $C_\infty \subset \partial \widehat{\text{AdS}}$.

Another example of a hyperbolic plane is the plane dual to an involution itself. For example, let $g \in \text{AdS}^3$ be the involution $\mathbf{J}(0)$ defined in (11) and corresponding to $i \in \mathbb{H}^2$. Then one lift of g^* to $\widehat{\text{AdS}}$ is the unit disc D^2 in the xy -plane defined by $z = 0$. The other lift is the exterior of D^2 in the xy -plane, which is the image $I_{\mathcal{S}}(D^2)$. The closure of each disc intersects \mathcal{S} in the unit circle in the xy -plane.

An example of a degenerate totally geodesic surface is the Borel subgroup $\mathfrak{B} = \exp(\mathfrak{b}) \subset \text{PSL}(2, \mathbb{R})$ consisting of the images of upper-triangular matrices in $\text{SL}(2, \mathbb{R})$; compare the discussion in §2.3.3. Lift \mathfrak{B} to the connected subgroup $\widehat{\mathfrak{B}} \subset \text{SL}(2, \mathbb{R})$, which embeds in Ein^3 by:

$$(27) \quad \begin{array}{c} \widehat{\mathfrak{B}} \xrightarrow{\Psi} \text{Ein}^3 \\ \left[\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right] \longmapsto [a - a^{-1} : b : b : (a + a^{-1}) - 2 : (a + a^{-1}) + 2]. \end{array}$$

The image is the subspace defined by $Y = Z$ but $U \neq V$. In affine coordinates (x, y, z) on \mathbf{E}^3 , this is the plane $y = z$. Its closure in \mathbf{Ein}^3 is the lightcone $L(p_1)$ where

$$p_1 = [0 : 1 : 1 : 0 : 0] \in C_\infty.$$

Furthermore the boundary of $\Psi(\widehat{\mathfrak{B}})$ consists of two rulings \mathcal{S}_0^+ (given by $x = 1$) and \mathcal{S}_π^- (given by $x = -1$), as defined in (20).

Another degenerate plane is defined by the subgroup \mathfrak{B}^\dagger consisting of the images of *lower triangular matrices*. It embeds via

$$(28) \quad \widehat{\mathfrak{B}}^\dagger \xrightarrow{\Psi} \mathbf{Ein}^3$$

$$\begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \mapsto [a - a^{-1} : c : -c : (a + a^{-1}) - 2 : (a + a^{-1}) + 2].$$

The image is the subspace defined by $Y = -Z$ but $U \neq V$. In affine coordinates (x, y, z) on \mathbf{E}^3 , this is the plane $y = -z$. Its closure in \mathbf{Ein}^3 is the lightcone $L(p_2)$ where

$$p_2 = [0 : 1 : -1 : 0 : 0] \in C_\infty.$$

Furthermore the boundary of $\Psi(\widehat{\mathfrak{B}})$ consists of two rulings \mathcal{S}_0^- (given by $x = 1$) and \mathcal{S}_π^+ (given by $x = -1$), as defined in (20).

Finally we consider an indefinite plane discussed in §2.3.2. It maps to \mathbf{Ein}^3 by:

$$(29) \quad \mathfrak{G} \xrightarrow{\Psi} \mathbf{Ein}^3$$

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix} \mapsto [0 : b + c : b - c : 2a - 2 : 2a + 2]$$

where $a^2 - bc = 1$. The image $\Psi(\mathfrak{G})$ is defined by $X = 0$ and $U \neq V$.

4. CROOKED SURFACES ADAPTED TO AN INVOLUTION

This section develops the theory of crooked surfaces, especially as they relate to an involution $I_{\mathcal{I}}$ in an Einstein hypersphere. The main goal is to show that classic AdS-crooked planes correspond to crooked surfaces in \mathbf{Ein}^3 which are adapted to $I_{\mathcal{I}}$.

A useful invariant of a crooked plane is its *spine*: the unique spacelike geodesic σ lying on the crooked plane. The spine contains the vertex p and the pair (p, σ) determines the crooked plane.

In a similar way, we define a *spine* $\bar{\sigma}$ for a crooked surface \mathfrak{C} . When \mathfrak{C} is the closure of a crooked plane $\mathcal{C} \in \mathbf{E}^3$, then $\bar{\sigma}$ equals the union $\sigma \cup \{p_\infty\}$, where p_∞ is the improper point. We relate this invariant to the parametrization of crooked surfaces by stem-configurations discussed in [2]. The spine $\bar{\sigma}$ is a spacelike circle containing the vertex p_0 and the

improper point p_∞ . In general triples $(\bar{\sigma}, p_0, p_\infty)$, where $\bar{\sigma}$ is a spacelike circle in Ein^3 and $p_0, p_\infty \subset \bar{\sigma}$ are distinct points, provide an alternate parametrization of crooked surfaces.

We say that \mathfrak{C} is *adapted* to $I_{\mathcal{J}}$ if and only if its spine $\bar{\sigma}$ is invariant under $I_{\mathcal{J}}$ and p_0, p_∞ are interchanged under $I_{\mathcal{J}}$.

4.1. Crooked surfaces. A *crooked surface* is a piecewise smooth hypersurface in Ein^3 constructed as follows. Take two points q_0, q_∞ in Ein^3 which do not lie on a photon. Call one of them, say q_0 , the *vertex*. Call the other one, q_∞ , the *covertex*. The intersection of the lightcones $L(q_0) \cap L(q_\infty)$ is a spacelike circle in Ein^3 disjoint from $\{q_0, q_\infty\}$. Choose two distinct points $q_1, q_2 \in \text{Ein}^3$ which lie on $L(q_0) \cap L(q_\infty)$. Call the points q_1, q_2 the *hingepoints* of the crooked plane.

For $i = 0, \infty$ and $j = 1, 2$, let ϕ_i^j be the unique photon containing q_i and q_j . Call the photons ϕ_0^1, ϕ_0^2 *hinges* and the photons $\phi_\infty^1, \phi_\infty^2$ *cohinges*. Call such a quadruple $(q_0, q_\infty; q_1, q_2)$ a *stem configuration*. Such a configuration lies on a unique Einstein hypersphere W .

The hinges and cohinges are rulings of W and decompose W into four squares, each of which have the same four vertices q_0, q_1, q_∞, q_2 . Two of these squares contain timelike directions emanating from q_0 or q_∞ , and constitute the two components T_1, T_2 of the *stem*.

The *wings* are obtained from the lightcones $L(q_j)$ as follows. The photons ϕ_0^j, ϕ_∞^j separate $L(q_j)$ into two bigons, one of which attaches to $T_1 \cup T_2$.

4.1.1. The spine of a crooked surface. Spacelike circles in Ein^3 correspond to projectivized nullcones of three-dimensional subspaces $F \subset \mathbb{R}^{3,2}$ of signature $(2, 1)$. That is, F is the image of an isometric embedding $\mathbb{R}^{2,1} \hookrightarrow \mathbb{R}^{3,2}$ of inner product spaces. Its orthogonal complement F^\perp is nondegenerate as well, and has signature $(1, 1)$ and together F, F^\perp define an orthogonal decomposition

$$\mathbb{R}^{2,1} \oplus \mathbb{R}^{1,1} \xrightarrow{\cong} F \oplus F^\perp = \mathbb{R}^{3,2}.$$

The projectivized null cone of $\mathbb{R}^{1,1}$ defines an unordered pair of nonincident points in Ein^3 . Orthogonal complementation defines a natural bijection between spacelike circles and unordered pairs of nonincident points. When applied to such a pair $\{p_1, p_2\} \subset \text{Ein}^3$, the corresponding spacelike circle equals the intersection of lightcones $L(p_1) \cap L(p_2)$.

Suppose that $\mathfrak{C} = \mathfrak{C}(p_0, p_\infty; p_1, p_2)$ is a crooked surface. Define the *spine* $\bar{\sigma}$ of \mathfrak{C} as the spacelike circle dual to the pair of hingepoints p_1, p_2 :

$$\bar{\sigma} := L(p_1) \cap L(p_2).$$

Since p_0 and p_∞ are each incident to p_i for $i = 1, 2$, both p_0 and p_∞ lie in $\bar{\sigma}$. The triple $(p_0, p_\infty; \bar{\sigma})$ is equivalent to the stem-configuration $(p_0, p_\infty; p_1, p_2)$ and equally well determines the crooked surface \mathfrak{C} .

4.1.2. *Invariant crooked surfaces and adapted crooked surfaces.* Suppose that \mathfrak{C} is a crooked surface invariant under the involution $I_{\mathcal{S}}$. Then $I_{\mathcal{S}}$ preserves the stem-configuration and permutes the pairs p_0, p_∞ and p_1, p_2 . The points p_0, p_∞ are characterized by the property that the tangent cones of \mathfrak{C} at these points are crooked planes. We say that \mathfrak{C} is *adapted* to $I_{\mathcal{S}}$ if $I_{\mathcal{S}}$ interchanges p_0 and p_∞ but fixes p_1 and p_2 .

This condition is stronger than being invariant. The only spacelike circles C invariant under $I_{\mathcal{S}}$ meet \mathcal{S} in two points. Thus the restriction of $I_{\mathcal{S}}$ is an orientation-reversing involution of C fixing this pair. In particular $I_{\mathcal{S}}$ -invariant crooked surfaces exist where $I_{\mathcal{S}}$ do not fix the hinge-points, and these will *not* be adapted in our definition.

4.2. **Completing $\widehat{\text{AdS}}$ crooked planes to crooked surfaces.** Next we show ((2) of the Main Theorem) that an AdS -crooked plane lifts to an object in $\widehat{\text{AdS}}$ whose closure in Ein^3 is a crooked surface, as defined in [11, 2, 16]. Call the special point corresponding to the improper point in a crooked surface arising from a crooked plane in Minkowski space) the *covertex*. As shown in [2], the vertex and covertex are related by an automorphism of Ein^3 which preserves the crooked surface. The condition that the crooked surface in Ein^3 arises from an AdS -crooked plane is that the vertex and covertex are related by the involution $I_{\mathcal{S}}$.

Begin with an AdS -crooked plane $C(g, \mathbf{s})$ as in §2.4. Working in the $\text{PSL}(2, \mathbb{R})$ -model, we may assume that the vertex g equals the identity element $e \in \text{PSL}(2, \mathbb{R})$.

Now lift the crooked plane to an $\widehat{\text{AdS}}$ -crooked plane $\widehat{C}(\hat{p}, \mathbf{s})$ in $\widehat{\text{AdS}}$ as in § 2.5.1. Then the vertex \hat{g} is the identity matrix $\mathbf{1} \in \text{SL}(2, \mathbb{R})$. The covertex is $-\mathbf{1} \in \text{SL}(2, \mathbb{R})$, the image of $\mathbf{1}$ under the deck involution. We identify the tangent spaces $T_e(\text{AdS}^3)$ with $T_1(\widehat{\text{AdS}})$ and denote the unit spacelike vector in $T_1(\widehat{\text{AdS}})$ corresponding to \mathbf{s} by $\hat{\mathbf{s}}$.

Under the embedding Ψ of $\widehat{\text{AdS}}$ in Ein^3 , the vertex $\mathbf{1}$ corresponds to the origin $0 = (0, 0, 0)$ in Minkowski space \mathbb{E}^3 and $\hat{\mathbf{s}}$ corresponds to a unit spacelike vector (still denoted $\hat{\mathbf{s}}$) in $T_0\mathbb{E}^3 \cong \mathbb{V}$. By applying yet another isometry of AdS^3 , we can assume that $\hat{\mathbf{s}}$ is dual to the geodesic $\mathbb{H}^1 \subset \mathbb{H}^2$ defined by (11), corresponding to the matrices in $\text{SL}(2, \mathbb{R})$ with zero diagonal entries.

Now we build up the picture of the $\widehat{\text{AdS}}$ -crooked plane and show that its closure in Ein^3 is the crooked surface which is the closure of a standard crooked plane in \mathbf{E}^3 . The main difference are the parametrizations of particles and the spine, since these non-null geodesics are complete in both the realization in Minkowski space and anti-de Sitter space. In Minkowski space these geodesics limit to the improper point p_∞ while in AdS^3 these geodesics limit to $\text{Ein}^2 = \mathcal{S} \cup \mathcal{J}$.

4.2.1. *Particles and the stem.* We begin with a particle, that is a time-like geodesic, and show that it is just a reparametrized particle in Minkowski space compactified by the improper point p_∞ . The stem is a union of particles, whose intersection points with p^* form a geodesic in the hyperbolic plane p^* . As in §3.5, the two lifts of p^* correspond to the components of the 2-sheeted hyperboloid $x^2 + y^2 - z^2 = -1$. The geodesic $\sigma \subset p^*$ corresponding to $\mathbf{H}^1 \subset \mathbf{H}^2$ is represented by the hyperbola

$$x = y^2 - z^2 + 1 = 0.$$

The stem was defined in §2.4 as the union of the timelike geodesics joining p to $\sigma \subset p^*$. Parametrizing the geodesic $\sigma \subset \mathbf{H}^2$ by $\mathbf{J}(t)$ (where $t \in \mathbb{R}$), as in (11), the corresponding elliptic one-parameter subgroups are:

$$\mathcal{E}(\theta, t) := \cos(\theta)\mathbf{1} + \sin(\theta)\mathbf{J}(t) = \begin{bmatrix} \cos(\theta) & -\sin(\theta)e^t \\ \sin(\theta)e^{-t} & \cos(\theta) \end{bmatrix}$$

we obtain the stem of an AdS -crooked plane \mathbf{C} . The corresponding subset of $\text{SL}(2, \mathbb{R})$ is defined by $a = d$ with $-2 < a = d < 2$. Compare (12) and §2.3.2.

Under the embedding $\widehat{\text{AdS}} \hookrightarrow \text{Ein}^3$ in (19), this is an open subset in the hypersurface $X = 0$, that is the subset defined by:

$$x = 0, \quad y^2 - z^2 < 0,$$

which defines the stem of the standard example of a crooked plane, as in [9]. The embedding of this totally geodesic submanifold of $\widehat{\text{AdS}}$ is explicitly given in (29).

4.2.2. *Hinges and cohinges.* The stem is bounded by two hinges, which in AdS^3 is the pair of unipotent one-parameter subgroups

$$(30) \quad \begin{aligned} U_1 &:= \pm \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} \\ U_2 &:= \pm \left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} \end{aligned}$$

which respectively correspond to the *hinges* $h_1, h_2 \subset \text{AdS}^3$. By (30), their $\widehat{\text{AdS}}$ -lifts map via Ψ to arcs of photons

$$(31) \quad \begin{aligned} \hat{h}_1^+ &:= \left\{ [0 : t : t : 0 : 4] \mid t \in \mathbb{R} \right\} \\ \hat{h}_1^- &:= \left\{ [0 : t : t : 4 : 0] \mid t \in \mathbb{R} \right\} \\ \hat{h}_2^+ &:= \left\{ [0 : t : -t : 0 : 4] \mid t \in \mathbb{R} \right\} \\ \hat{h}_2^- &:= \left\{ [0 : t : -t : 4 : 0] \mid t \in \mathbb{R} \right\} \end{aligned}$$

in homogeneous coordinates, respectively. The hinges \hat{h}_i of the tangent cone \mathcal{C} exponentiate to the hinges of the $\widehat{\text{AdS}}$ crooked plane $\widehat{\mathcal{C}}$. Since the exponential map is linear (with speed $1/4$), the hinges of $\widehat{\mathcal{C}}$ are also complete geodesics in $\widehat{\text{AdS}}$. Their closures consist of their unions with hinge points which lie on C_∞ . The co-hinges \hat{h}_i^- are just the images of the hinges under inversion $I_{\mathcal{I}}$, and their closures also consist of their unions with hinge points:

$$\begin{aligned} \partial \hat{h}_1^+ &= \partial \hat{h}_1^- = p_1 = [0 : 1 : 1 : 0 : 0] \\ \partial \hat{h}_2^+ &= \partial \hat{h}_2^- = p_2 = [0 : 1 : -1 : 0 : 0], \end{aligned}$$

respectively. In Minkowski space these hinges are just the lines consisting of points $(0, y, z) \in \mathbb{E}^3$ where $y = z$ and $y = -z$ respectively. and the co-hinges are their $I_{\mathcal{I}}$ -images in the inverse cone \mathcal{I} . The hinge points $p_i = \partial \hat{h}_i^\pm$ (for $i = 1, 2$) each lie on the ideal circle C_∞ .

4.2.3. *Wings and cowings.* From the hinges h_1, h_2 we determine the halfplanes h_i^\perp (where $i = 1, 2$) in Minkowski space. These will intersect $\widehat{\text{AdS}}$ in totally geodesic null surfaces, and the complements of the hinges in these planes will be halfplanes, one of which will be the wings of the crooked plane. As the hinges h_\pm correspond to unipotent one-parameter subgroups U_\pm (as in (30) in §4.2.2), the associated null planes h_\pm^\perp correspond to the normalizers of these subgroups $B_\pm := \text{Norm}(U_\pm)$:

$$(32) \quad \pm \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}, \pm \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix}$$

which define totally geodesic surfaces in $\widehat{\text{AdS}}$ which are degenerate. The complement $h_\pm^\perp \setminus h_\pm$ is a disjoint union of two null halfplanes.

The unipotent one-parameter subgroup U (which is the commutator subgroup of B_{\pm}) is defined by $a = 1$. While U is the stabilizer of the vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2,$$

its normalizer B is the stabilizer of the corresponding point (denoted ∞) in the projective line $\mathbb{P}(\mathbb{R}^2)$. Furthermore U separates B into two subsets depending on whether $a > 1$ or $a < 1$. The complement $B \setminus U$ consists of hyperbolic elements in B , and the two subsets are distinguished by whether ∞ is an attracting (respectively repelling) fixed point.

The corresponding subsets in Ein^3 are, respectively:

$$[a - a^{-1} : b : b : (a + a^{-1}) - 2 : (a + a^{-1}) + 2]$$

$$[a - a^{-1} : c : -c : (a + a^{-1}) - 2 : (a + a^{-1}) + 2]$$

which are defined in homogeneous coordinates by $Y = Z$ and $Y = -Z$ respectively, and in affine Minkowski coordinates by $y = z$ and $y = -z$ respectively.

Consider the first subset $Y = Z$ which contains the hinge $X = U = Y - Z = 0$. The complement of the hinge in the wing is a pair of null half-planes, each of which intersects AdS^3 in a strip bounded by the hinge and a null line which is a ruling of \mathcal{S} .

4.2.4. The spine. As the wings in our examples correspond to subsemi-groups of Borel subgroups consisting of upper- and lower-triangular matrices, respectively, the spine consists of the Cartan subgroup of diagonal matrices. In particular (25) in §3.5.1 describes the mapping from the spine to the interval (23) on the x -axis, and (26) describes the mapping from the cospine to the union of two rays (24) on the x -axis, separated by the improper point at infinity. The spine $\bar{\sigma}$ of the corresponding crooked plane is then the x -axis completed by the improper point.

4.3. Characterization of AdS-crooked planes. Finally we prove the converse statement (3) of the Main Theorem.

Let $(q_0, q_{\infty}; q_1, q_2)$ be a stem configuration and

$$\mathfrak{C} := \mathfrak{C}(q_0, q_{\infty}; q_1, q_2) \subset \text{Ein}^3$$

the corresponding crooked surface. Suppose that \mathfrak{C} is $I_{\mathcal{S}}$ -adapted. Then $I_{\mathcal{S}}$ interchanges the vertex q_0 and the covertex q_{∞} and fixes the hinge-points q_1 and q_2 .

Then, since $I_{\mathcal{S}}$ fixes neither q_0 nor q_{∞} , both points lie in $\widehat{\Psi(\text{AdS})}$. Since the group of isometries $\text{Isom}(\widehat{\text{AdS}})$ acts transitively on $\widehat{\text{AdS}}$, we

may apply an automorphism of AdS^3 to assume that the covertex of \mathbf{C} is the improper point, that is, $q_\infty = p_\infty$. Since $I_{\mathcal{I}}$ interchanges the origin and the improper point, and interchanges the covertex and the vertex (because \mathbf{C} is $I_{\mathcal{I}}$ -adapted), the vertex equals the origin: $q_0 = p_0$. Because the covertex of \mathfrak{C} equals p_∞ , the crooked surface is the closure of a crooked plane \mathcal{C} in Minkowski space:

$$\mathcal{C} = \mathfrak{C} \setminus L(p_\infty) = \mathfrak{C} \cap \mathbb{E}^3$$

and the vertex of this crooked plane is the origin p_0 .

Now we adjust the hinge points. The group of orientation-preserving isometries $\widehat{\text{AdS}}$ which fix $\mathbf{1}$ and $-\mathbf{1}$ is just the group H of inner automorphisms of $\text{SL}(2, \mathbb{R})$, which is isomorphic to the identity component of $\text{SO}(2, 1)$. Similarly the group of orientation-preserving isometries of \mathbb{E}^3 which fix the origin equals $\text{SO}(2, 1)$, and the embedding Ψ is equivariant with respect to the isomorphism

$$\text{Inn}(\text{SL}(2, \mathbb{R})) \xrightarrow{\cong} \text{SO}(2, 1)^0 \cong \text{Isom}(\mathbb{H}^2)^0.$$

Since $\text{Isom}(\mathbb{H}^2)^0$ acts transitively on geodesics in \mathbb{H}^2 , the group $\text{SO}(2, 1)$ acts transitively on ordered pairs of distinct null lines in $\mathbb{R}^{2,1}$. Therefore there exists an isometry of $\widehat{\text{AdS}}$ which maps the hinge points q_i to p_i for $i = 1, 2$ respectively. Apply this isometry to assume that $q_i = p_i$. Thus \mathfrak{C} is the closure of the double cover $\widehat{\mathbf{C}}$ of an AdS -crooked plane \mathbf{C} . The proof is complete.

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