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Moduli spaces of local systems and higher Teichmüller theory. (English summary)
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This ambitious paper develops the theory of higher Teichmüller spaces over a compact connected oriented surface S with possibly nonempty boundary and punctures. These spaces generalize the classical Fricke-Teichmüller spaces whose points parametrize isometry classes of complete hyperbolic geometry structures on S, possibly with geodesic boundary.

Higher Teichmüller spaces originate with the moduli spaces $\mathcal{L}_{G,S}$ of G-local systems over S, where G is a connected $\mathbb{R}$-split semisimple algebraic Lie group. Such a local system is equivalent to a flat principal G-bundle over S. This, in turn, is equivalent to a conjugacy class of a representation $\pi_1(S) \xrightarrow{\rho} G$. The set $\mathrm{Hom}(\pi_1(S), G)$ has a natural structure of an affine algebraic set over $\mathbb{R}$, and $\mathcal{L}_{G,S}$ is its quotient (in the classical topology) by inner automorphisms of G.

The higher Teichmüller spaces involve some extra structure over the boundary, and correspond to a remarkable class of representations. Let B denote a Borel subgroup (minimal parabolic subgroup) of G and U its unipotent radical. A framing of a G-local system is a parallel section of the associated flat G/B-bundle over ∂S. A decoration of a G-local system is a parallel section of the associated flat G/U-bundle over ∂S, provided that the holonomy around each component of the boundary is unipotent. Equivalently a framing (respectively a decoration) corresponds to, for each component $\gamma \subset \partial S$, an element of G/B (respectively G/U) invariant under $\rho(\gamma)$. The authors define moduli spaces $\mathfrak{X}_{G,S}$ of framed local systems and $\mathcal{A}_{G,S}$ of decorated local systems, each of which maps to $\mathcal{L}_{G,S}$. They conjecture that these two moduli spaces are dual when the group G is replaced by its Langlands dual. Specifically, this means that there is a basis of functions on the framed moduli space $\mathfrak{X}_{G,S}$ parametrized by the points of the tropicalization of the decorated moduli space $\mathcal{A}_{G,S}$, and vice versa.

The paper develops a structure on these moduli spaces similar to that of a toric variety, whereby the space admits a dense open subset which looks like an affine torus $(\mathbb{G}_m)^N$ with a natural "symplectic geometry". When $\partial S = \varnothing$, this is the symplectic geometry given by the general construction in [W. M. Goldman, Adv. in Math. **54** (1984), no. 2, 200–225; MR0762512 (86i:32042)], which was based on [M. F. Atiyah and R. H. Bott, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615; MR0702806 (85k:14006)] for the analogous case that G is compact. In the simplest case ($G = \mathrm{PSL}(2,\mathbb{R})$), all three moduli spaces are the Fricke-Teichmüller space $\mathfrak{T}(S)$ of marked hyperbolic structures on S, the moduli space in question, and the symplectic structure arises from the Kähler form of the Weil-Petersson metric on Teichmüller space.

When $\partial S \neq \varnothing$, the moduli space $\mathcal{A}_{G,S}$ of decorated local systems carries a natural degenerate closed exterior 2-form. Its "dual" moduli space $\mathfrak{X}_{G,S}$ of framed local systems carries a natural Poisson structure on $\mathcal{L}_{G,S}$. In the analogous case when G is compact, these structures relate to the

Poisson structures on moduli spaces considered by K. Guruprasad et al. [Duke Math. J. **89** (1997), no. 2, 377–412; MR1460627 (98e:58034)] and the quasi-Hamiltonian moment maps considered by A. Yu. Alekseev, A. Z. Malkin and E. Meinrenken [J. Differential Geom. **48** (1998), no. 3, 445–495; MR1638045 (99k:58062)]. Closely related are the constructions, using quantum groups, of V. V. Fok and A. A. Roslyĭ [in *Moscow Seminar in Mathematical Physics*, 67–86, Amer. Math. Soc. Transl. Ser. 2, 191, Amer. Math. Soc., Providence, RI, 1999; MR1730456 (2001k:53167)] and Alekseev and Malkin [Comm. Math. Phys. **169** (1995), no. 1, 99–119; MR1328263 (96m:58028)]

The locally toric structure on these moduli spaces has a remarkable property: one component of this space has a positive structure. From the viewpoint of the paper under review, the disconnectedness of the space $\mathrm{Hom}(\pi_1(S), G)$ (corresponding to the moduli space $\mathcal{L}_{G,S}$ of local systems) arises from the disconnectedness of the Lie group $\mathbb{R}^*$.

In the simplest case ($G = \mathrm{SL}(2, \mathbb{R})$), the Fricke-Teichmüller space $\mathfrak{T}(S)$ of marked complete hyperbolic structures on S embeds as a connected component in the moduli space, defined by real inequalities (conditions like $|\mathrm{tr}(\rho(\gamma))| \geq 2$, for example). The dense open affine torus $(\mathbb{G}_m)^N$ then corresponds to the shearing coordinates developed by W. Thurston and by R. C. Penner [Comm. Math. Phys. **113** (1987), no. 2, 299–339; MR0919235 (89h:32044)]. Since this plays a fundamental role in this theory, we briefly review the construction.

The starting point for this theory is an ideal triangulation of S and the shearing coordinates on $\mathfrak{T}(S)$ first studied in this context by Thurston and Penner. The surface S, with a convex hyperbolic structure, is decomposed into ideal polygons. (The first occurrence of this idea seems to be in the 1983 doctoral thesis of Lee Mosher ["Pseudo-Anosovs on punctured surfaces", Princeton Univ., Princeton, NJ, 1983] written under the supervision of Thurston.) When S has cusps, then the sides of the polygons may be simple geodesics which limit to the cusps. When $\partial\overline{S} \neq \varnothing$, then $\partial\overline{S}$ is assumed to be a union of closed geodesics, and the sides of the polygon may spiral around these closed geodesics, or other closed geodesics in the interior of S. The surface is reconstructed from this finite set of polygons by identifying sides (which appear as shears), and these gluing instructions furnish a convenient and computable set of coordinates for $\mathfrak{T}(S)$. As first observed by Penner [J. Differential Geom. **35** (1992), no. 3, 559–608; MR1163449 (93d:32029)], the Weil-Petersson symplectic form has a remarkably simple expression in these coordinates. Penner's construction is based in turn on S. A. Wolpert's theorem that Fenchel-Nielsen coordinates on $\mathfrak{T}(S)$ are canonical (Darboux) coordinates for the Weil-Petersson Kähler form [S. A. Wolpert, Amer. J. Math. **107** (1985), no. 4, 969–997; MR0796909 (87b:32040)].

The shearing coordinates provide instructions to assemble a hyperbolic surface out of ideal 2-simplices. The condition that the shear coordinates are positive implies that the union of ideal 2-simplices fit together to form a nonsingular hyperbolic surface. Otherwise the union is a hyperbolic surface folded along the geodesic 1-simplices. These correspond to representations in other components of $\mathrm{Hom}(\pi_1(S), G)$, and have been investigated by R. M. Kashaev [Math. Res. Lett. **12** (2005), no. 1, 23–36; MR2122727 (2005k:53164)].

Such ideal triangulations are related by sequences of mutations, whereby one edge is removed and replaced by a geodesic with an alternate pair of endpoints, such as replacing one diagonal in a quadrilateral by the other diagonal. The coordinates transform birationally, preserving the symplectic geometry and the positivity. They define a groupoid, whose objects are ideal triangulations, and the morphisms are generated by the elementary moves. Because $\mathrm{Mod}(S)$ has only finitely many orbits on the set of ideal triangulations, this group is a finite extension of $\mathrm{Mod}(S)$.

Fok and Goncharov show that this theory extends to all split real forms G. In their generalized shearing coordinates, the elementary transformations are represented by rational functions whose numerators and denominators are polynomials whose coefficients are positive integers. Therefore inside the coordinate ring of the moduli space is a preserved subset of positive functions. Moreover this positive structure on the moduli space determines a preferred subset of positive points, which comprises a connected component in the classical topology of the set of $\mathbb{R}$-points. From its description this component is homeomorphic to a cell $\mathbb{R}^N$. This positivity was due to G. Lusztig in his theory of canonical bases [in *Lie theory and geometry*, 531–568, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994; MR1327548 (96m:20071); in *Algebraic groups and Lie groups*, 281–295, Cambridge Univ. Press, Cambridge, 1997; MR1635687 (2000j:20089)], and independently to A. Zelevinsky.

The authors describe this in the general algebraic framework they call an orbi-cluster ensemble. The ideal triangulations correspond to the seeds and the mutations which closely relate to the cluster algebras developed by S. Fomin and Zelevinsky [J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic); MR1887642 (2003f:16050); Invent. Math. **154** (2003), no. 1, 63–121; MR2004457 (2004m:17011); Adv. in Appl. Math. **28** (2002), no. 2, 119–144; MR1888840 (2002m:05013)]. As noted in the paper the relationship between cluster algebras and Penner's Weil-Petersson symplectic geometry on decorated Teichmüller spaces was independently discussed by M. I. Gekhtman, M. Z. Shapiro and A. Vainshtein [Duke Math. J. **127** (2005), no. 2, 291–311; MR2130414 (2006d:53103); correction, Duke Math. J. **139** (2007), no. 2, 407–409; MR2352136 (2008f:53110)]. While the mutations for higher groups appear the same as for $\mathrm{SL}(2)$ the expression of flips becomes increasingly complicated—for example for SL(3) flips require four mutations.

Furthermore the action of the mapping class group $\mathrm{Mod}(S)$ on these spaces preserves all this structure. Starting from the Poisson structure, the authors then develop a quantization of this space, from which new actions and extensions of the mapping class group derive. This generalizes earlier work in this direction by Fok and L. O. Chekhov [Teoret. Mat. Fiz. **120** (1999), no. 3, 511–528; MR1737362 (2001g:32034)]. Extending the mapping class group of a surface to a groupoid generated by flips appears in earlier work of Penner [Adv. Math. **98** (1993), no. 2, 143–215; MR1213724 (94k:32032); in *Geometric Galois actions, 2*, 293–312, Cambridge Univ. Press, Cambridge, 1997; MR1653016 (99j:32024)]. These quantum representations of the mapping class were an important motivation for this study, on which the authors have recently made progress [Invent. Math. **175** (2009), no. 2, 223–286; MR2470108].

The symplectic form is also described in terms of the algebraic K-theory of the moduli space $\mathcal{A}_{G,S}$ of decorated local systems. The (possibly degenerate) closed 2-form defines an element of K_2 of the function field of $\mathcal{A}_{G,S}$. Its explicit description [see A. B. Goncharov, in *I. M. Gel′fand Seminar*, 169–210, Amer. Math. Soc., Providence, RI, 1993; MR1237830 (95c:57045)] displayed the canonical coordinate systems, which initiated this investigation. As Fok has pointed out to the reviewer, pursuing this approach relates K_3 of this field with volumes of simplices in the symmetric space.

The positive structure allows one to tropicalize this variety. In the simplest case, the tropical points identify with measured geodesic laminations, whose projectivizations comprise Thurston's boundary for $\mathfrak{T}(S)$. The relation between Thurston's symplectic form on the measured lamination space and the Weil-Petersson Kähler form is due to A. Papadopoulos and Penner [Trans. Amer. Math. Soc. **335** (1993), no. 2, 891–904; MR1089420 (93d:57022); C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 11, 871–874; MR1108510 (92e:57023)]. That Thurston's spaces tropicalize the real character variety is implicitly due to J. W. Morgan and P. B. Shalen [Ann. of Math. (2) **120** (1984), no. 3, 401–476; MR0769158 (86f:57011)] and is related to George Bergman's logarithmic limit set of an affine variety [Trans. Amer. Math. Soc. **157** (1971), 459–469; MR0280489 (43 #6209)].

For other G, this defines a new structure, which deserves further study. In particular the extension of Thurston's theory of measured laminations on hyperbolic surfaces (such as train track coordinates, earthquake deformations, bending, cataclysms) to higher Teichmüller theory raises many fascinating questions. The paper under review treats the case of $\mathrm{SL}(n)$, but for the other split real forms, the reader should consult the authors' sequel [in *Algebraic geometry and number theory*, 27–68, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006; MR2263192 (2008b:22009)], but the cluster theory for general G is not given here.

When $G = \mathrm{SL}(3, \mathbb{R})$ this is the deformation space of convex $\mathbb{R}\mathrm{P}^2$-structures on S, which was discussed in the authors' shorter paper [Adv. Math. **208** (2007), no. 1, 249–273; MR2304317 (2008g:57015)]. For compact surfaces this deformation space was shown to be a cell when S is a compact surface with boundary by the reviewer [J. Differential Geom. **31** (1990), no. 3, 791–845; MR1053346 (91b:57001)].

In general, the higher Teichmüller space coincides with the Teichmüller component (now called the Hitchin component) of the space $\mathrm{Hom}(\pi, G)/G$ discovered by Nigel Hitchin [Topology **31** (1992), no. 3, 449–473; MR1174252 (93e:32023)]. Using gauge-theoretic techniques and a complex structure J on S, Hitchin identified a connected component of $\mathrm{Hom}(\pi, G)/G$ with the complex vector space of sections of a holomorphic vector bundle over the Riemann surface (S, J). F. Labourie [Invent. Math. **165** (2006), no. 1, 51–114; MR2221137 (2007c:20101)] discovered strong dynamical properties of the representations in Hitchin's component, and proved that such representations quasi-isometrically embed $\pi_1(S)$ in G, and in particular define isomorphisms of $\pi_1(S)$ with discrete subgroups of G. O. Guichard [J. Differential Geom. **80** (2008), no. 3, 391–431; MR2472478 (2009h:57031)] completed Labourie's characterization of these representations. Specifically the curve $S^1 \xrightarrow{f} \mathbb{P}^n$ is hyperconvex if for every collection $x_0, \ldots, x_n \in S^1$ consisting of distinct points, the lines in $\mathbb{R}^{n+1}$ corresponding to $f(x_0), \ldots, f(x_n)$ span $\mathbb{R}^{n+1}$. Crucial to this point of view is that the limit set of these groups is positive in the above sense; Labourie established that these positive curves are Hölder regular, which is an important feature in this theory.

Among the many intriguing questions raised in this paper is whether a representation $\pi_1(S) \longrightarrow G_{\mathbb{C}}$ (where $G_{\mathbb{C}}$ is the group of $\mathbb{C}$-points) which is close to a Hitchin representation in G determines a pair of Hitchin representations into G, that is, a pair of points in the "higher Teichmüller space". The evidence for this conjecture is the classical case when $G = \mathrm{SL}(2, \mathbb{R})$, in which L. Bers's simultaneous uniformization for quasi-Fuchsian deformations of Fuchsian representations parametrizes quasi-Fuchsian surface groups [L. Bers, Bull. Amer. Math. Soc. **66** (1960), 94–97; MR0111834 (22 #2694)]. Bers's proof uses heavily the theory of quasiconformal mappings in dimension two, a tool which seems very difficult to extend to this more general setting where complicated integrability conditions are present.

Another provocative question arising from this theory is to what extent what the authors call "Weil-Petersson" is a mapping class group invariant Kähler geometry on the higher Teichmüller spaces.

Despite the length of the paper (211 pages), it is clearly written. The 30-page introduction is particularly helpful for an overview of the theory. Although parts of the paper are somewhat speculative, this paper contains a wealth of interesting new ideas and inter-relationships between several areas of mathematics. Undoubtedly this work will strongly impact and inspire future research.

Reviewed by *William Goldman*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.