Hyperbolizing Surfaces

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Outline

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- Group theory arises in topology through the *fundamental group* and the *universal covering space.*

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- Σ acquires *geometric structure M modeled* on (G, X). (Ehresmann 1936)

Development

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equivariant with respect to a *holonomy representation*

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- The space of *marked structures* on on a fixed topology Σ forms a *deformation space* locally modeled on $\mathrm{Hom}(\pi, G)/G$ (Thurston 1978):

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G, X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

$$\xrightarrow{\text{hol}} \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

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- Real projective structures (hyperbolic structures via Klein model, Euclidean, elliptic).

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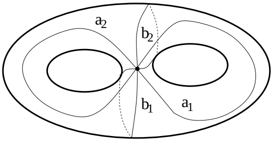
  - *Mapping class group*

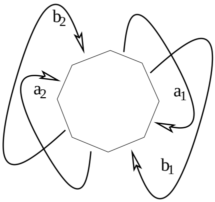
$$\mathrm{Mod}(\Sigma) = \pi_0\big(\mathrm{Diff}(\Sigma)\big) \cong \mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$$

acts on $\mathrm{Hom}(\pi, G)/G$.

The fundamental group of a closed surface

Obtain a genus g surface from a $4g$-gon.





A presentation for the fundamental group

- The fundamental group $\pi = \pi_1(\Sigma)$ is the fundamental group of a closed orientable surface admits a presentation

$$\pi = \langle A_1, \dots, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

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- A representation $\pi_1(\Sigma)$ in a group G is just

$$(\alpha_1, \dots, \beta_g) \in G^{2g}$$

satisfying the *defining relation*

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = 1.$$

Navigating the deformation space

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$$\rho_t : \begin{cases} A_i & \longmapsto \rho(A_i) \text{ if } i \geq 1 \\ B_j & \longmapsto \rho(B_j) \text{ if } j > 1 \\ B_1 & \longmapsto \rho(B_1)\zeta(t) \end{cases}$$

where $\zeta(t)$ path in the centralizer of $\rho(A_1)$.

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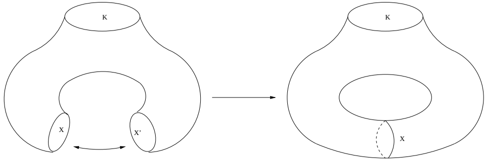
  - *Fenchel-Nielsen twist flow:* ($G = SL(2,\mathbb{R})$): $\zeta(t)$ group of *transvections* along $\rho(A_1)$-invariant geodesic in H^2 —



Observing the deformation space

- A natural class of functions on $\mathrm{Hom}(\pi, G)/G$ arise from functions $G \xrightarrow{f} \mathbb{R}$ invariant under conjugation and $\alpha \in \pi$:

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$$[\rho] \longmapsto f\big(\rho(\gamma)\big)$$

- The *trace* of any linear representation $G \longrightarrow GL(N, \mathbb{R})$
- The *geodesic displacement function* (only defined for hyperbolic elements)

$$\mathrm{tr}(\gamma) = \pm 2 \cosh\big(\ell(\gamma)/2\big)$$

if $\gamma \in SL(2, \mathbb{R})$ is hyperbolic.

Symplectic structure

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- $G = \mathbb{R}$ or $\mathbb{C} \Longrightarrow$ Hom(π, G) is a real (or complex) *symplectic vector space* $H^1(\Sigma)$.
- α represented by a simple closed curve on Σ,

$$\text{Inn}(G)\text{-invariant function } G \xrightarrow{f} \mathbb{R}$$

$\Longrightarrow$ Hamiltonian flow of f_α covered by *generalized twist flow* on Hom(π, G).

Fricke space

- Deformation space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures Σ corresponds to *discrete embeddings:*

$$\pi := \pi_1(\Sigma) \overset{\rho}{\hookrightarrow} \mathsf{PSL}(2, \mathbb{R})$$

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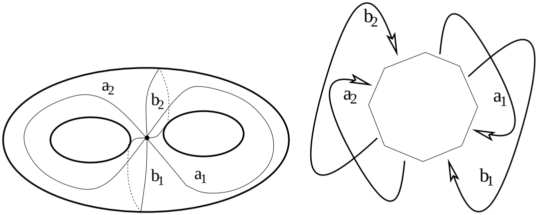
  - Equality $\Longleftrightarrow$ ρ is a discrete embedding. (1980)

Branched hyperbolic structures



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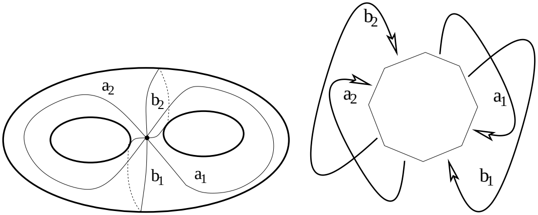

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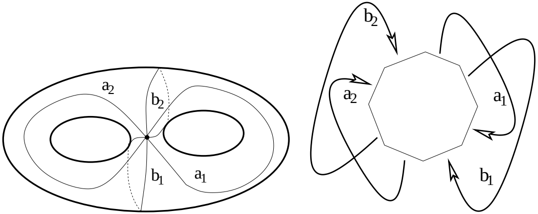

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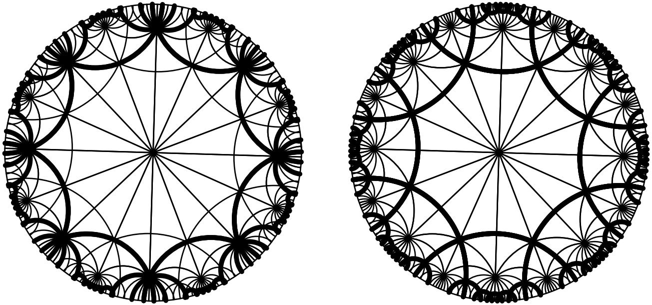
- Holonomy representation of a hyperbolic surface with cone angles $2\pi k_i$ extends to $\pi_1(\Sigma)$ with Euler number

$$e(\rho) = 2 - 2g + \sum k_i.$$

A hyperbolic surface of genus two



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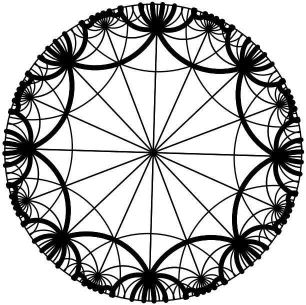

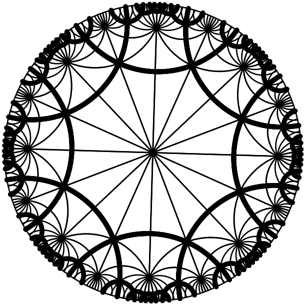

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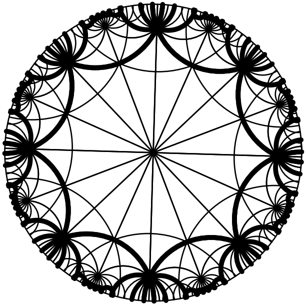

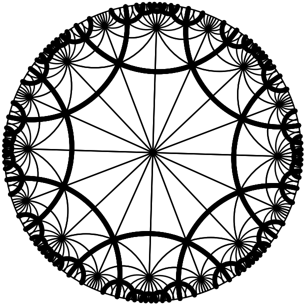

- Identifying a regular octagon with angles $\pi/4$ yields a nonsingular hyperbolic surface with $e(\rho) = \chi(\Sigma) = -2$.
- But when the angles are $\pi/2$, the surface has one singularity with cone angle 4π and

$$e(\rho) = 1 + \chi(\Sigma) = -1.$$

Branched hyperbolic structures

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- Each component of $\mathrm{Hom}(\pi, \mathrm{PSL}(2,\mathbb{R}))$ contains holonomy of branched hyperbolic structures.
- The Euler class $2-2g+k$ component deformation retracts onto k-fold symmetric product. (Hitchin 1987)

Dynamic/homotopic triviality

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  - $\mathrm{Mod}(\Sigma)$-invariant complex structure on $\mathfrak{F}(\Sigma)$.
  - For $G = \mathrm{PSL}(2,\mathbb{R})$, the general symplectic structure and the complex structure from Teichmüller space are part of the *Weil-Petersson* Kähler geometry on $\mathfrak{F}(\Sigma)$.

Quasi-Fuchsian groups

The group of orientation-preserving isometries of $\mathsf{H}^3_{\mathbb{R}}$ equals $\mathsf{PSL}(2,\mathbb{C})$. Close to Fuchsian representations in $\mathsf{PSL}(2,\mathbb{R})$ are *quasi-Fuchsian representations* $\mathcal{QF}$*:*

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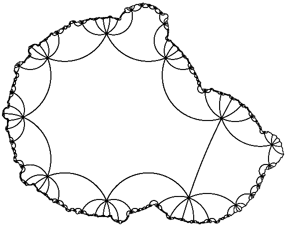

Complex hyperbolic geometry

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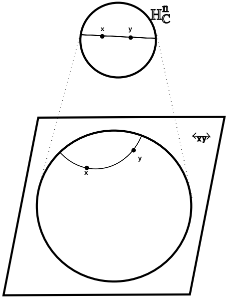

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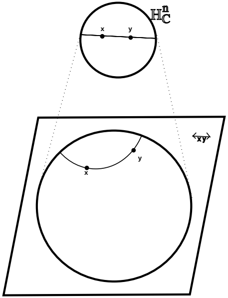

- $\mathbb{C}$- linear subspaces meet $H^n_{\mathbb{C}}$ in totally geodesic subspaces.

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- Components detected by a $\mathbb{Z}$-valued *characteristic* class generalizing the Euler class. (Toledo 1986, Xia 1997, Gothen 1997)

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- In general the analytic germ of a *reductive representation* of the fundamental group of a compact Kähler manifold is defined by a system of homogeneous quadratic equations. (G Millson 1988)
- For an $\mathsf{SU}(1,1)$-representation ρ_0, the neighborhood of

$$\pi \xrightarrow{\rho} \mathsf{SU}(1,1) \subset \mathsf{SU}(2,1)$$

in $\mathsf{Hom}(\pi, \mathsf{SU}(2,1))$ looks like the product of $\mathsf{Hom}(\pi, \mathsf{U}(1,1) \times \mathsf{U}(1))$ and a cone defined by a quadratic form of signature $e(\rho_0)$ on $\mathbb{R}^{4g-4}$.

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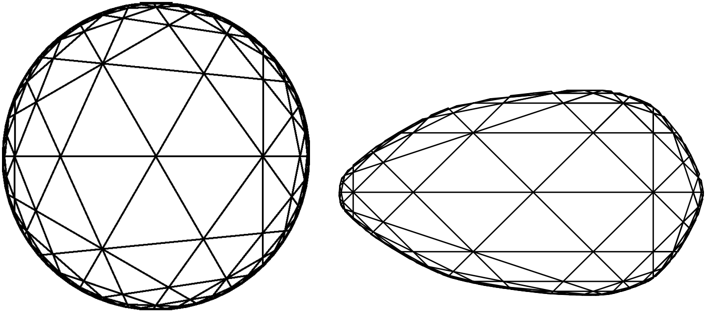
Deformations of triangle groups



Domains in $\mathbb{RP}^2$ tiled by $(3, 3, 4)$-triangles.

The deformation space of convex $\mathbb{RP}^2$-structures

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- $\mathfrak{C}(\Sigma)$ identifies with the holomorphic vector bundle over $\mathrm{Teich}(\Sigma)$ whose fiber over a marked Riemann surface X equals the vector space $H^0(X, (\kappa_X)^3)$ of *holomorphic cubic differentials*
  (Labourie 1997, Loftin 2001).

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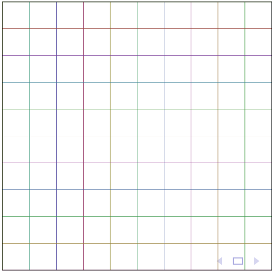

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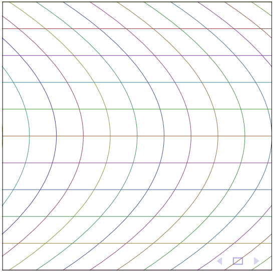

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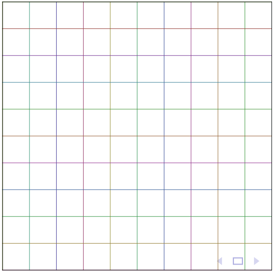

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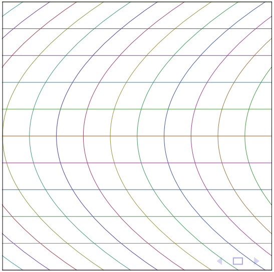

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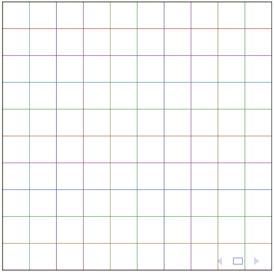

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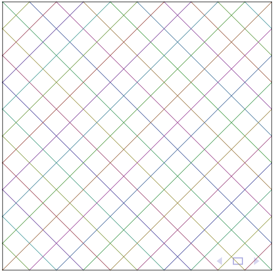

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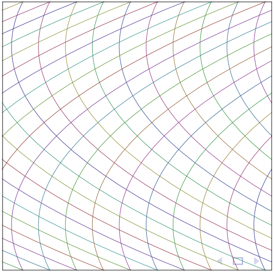

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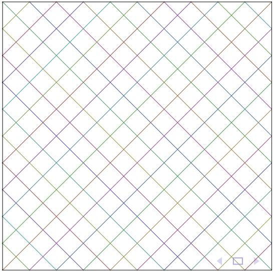

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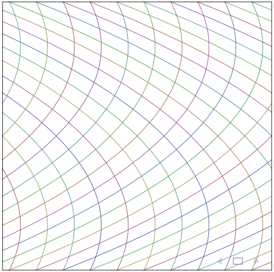

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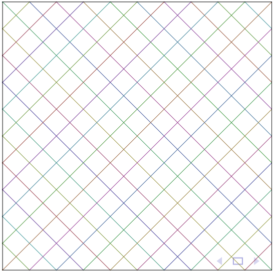

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- The orbit space — the *moduli space* of complete affine compact orientable 2-manifolds is non-Hausdorff.
- Contrast to the *proper* action of Mod(Σ) $\cong$ PGL$(2,\mathbb{Z})$ on $\mathfrak{F}(\Sigma)$ by *projective* transformations.