Deformations of geometric structures and representations of fundamental groups

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http://www.math.umd.edu/∼wmg/kaist.pdf

Enhancing Topology with Geometry

Deformations of geometric structure

Real projective structures

Representation varieties and character varieties

Hamiltonian flows of real projective structures

Geometry through symmetry

- In his 1872 *Erlangen Program*, Felix Klein proposed that a *geometry* is the study of properties of an abstract space X which are invariant under a transitive group G of transformations of X.
- Klein was heavily influenced by Sophus Lie, who was trying to develop a theory of *continuous groups*, to exploit *infinitesimal symmetry* to study differential equations, similar to how Galois exploited symmetry to study *algebraic* equations.

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Euclidean to affine to projective geometry

- Euclidean geometry: $X = \mathbb{R}^n$ Euclidean space and $G = \text{Isom}(X)$ the group of rigid motions:
- A *rigid motion* is a map $x \mapsto Ax + b$ where $A \in \text{O}(n)$ is orthogonal and $b \in \mathbb{R}^n$ is a translation vector.
- Invariant notions: Distance, angle, parallel, area, lines, ...
- Euclidean geometry: special case of *affine* geometry wheree $X = \mathbb{R}^n$ and $G = \text{Aff}(X)$, where $A \in \text{GL}(n, \mathbb{R})$ is only required to be *linear*.
- Only parallelism, lines preserved.
- Affine geometry: special case of *projective* geometry, when *parallelism* abandoned. $G = \text{PGL}(n+1, \mathbb{R})$, $X = \mathbb{RP}^n$.
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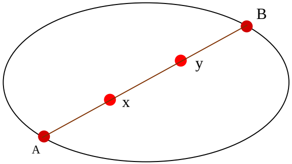
Other subgeometries of projective geometry

- Hyperbolic geometry: $X = \mathsf{H}^n \subset \mathbb{RP}^n$ $G = \mathsf{O}(n,1)$ the subset of $\mathsf{PGL}(n+1,\mathbb{R})$ stabilizing X;
- (Beltrami – Hilbert) Define the hyperbolic metric on X projectively in terms of cross-ratios:



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- More generally, one obtains a projectively invariant distance on any properly convex domain (Hilbert).

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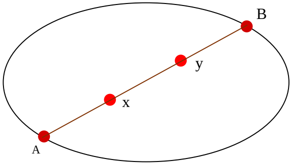

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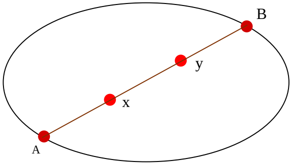

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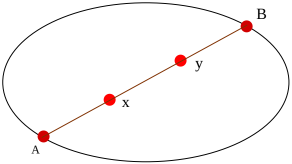

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Putting geometric structure on a topological space

- *Topology:* Smooth manifold Σ with coordinate patches U_α;
- Charts — *diffeomorphisms*

$$U_\alpha \xrightarrow{\psi_\alpha} \psi_\alpha(U_\alpha) \subset X$$

- For each component $C \subset U_\alpha \cap U_\beta$, $\exists g = g(C) \in G$ such that

$$g \circ \psi_\alpha|_C = \psi_\beta|_C.$$

- Local (G, X)-geometry defined by ψ_α independent of patch.
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Quotients of domains

- Suppose that $\Omega \subset X$ is an open subset invariant under a subgroup $\Gamma \subset G$ such that:
  - Γ is discrete;
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- Then $M = \Omega/\Gamma$ is a (G, X)-manifold
- The covering space $\Omega \longrightarrow M$ is a (G, X)-morphism.
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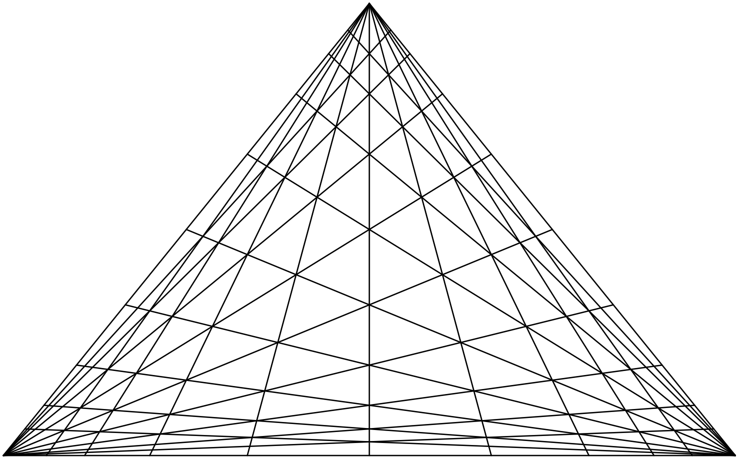
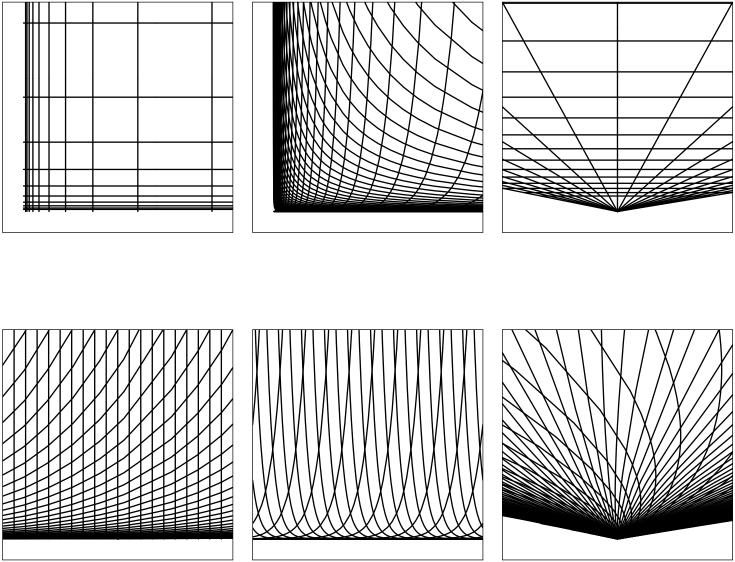
A projective $(3, 3, 3)$ triangle tesselation



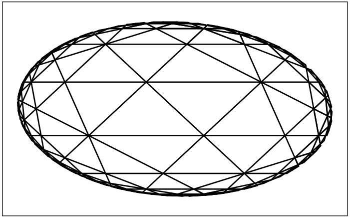
This tesselation of the open triangular region is equivalent to the tiling of the Euclidean plane by equilateral triangles.

Examples of incomplete quotient affine structures



Hyperbolic structures as $\mathbb{RP}^2$-structures

- Using the Klein-Beltrami model of hyperbolic geometry, the convex domain Ω bounded by a conic inherits a projectively invariant *hyperbolic geometry.*
- The charts for the hyperbolic structure determine charts for an $\mathbb{RP}^2$-structure.
- Every *hyperbolic manifold* is *convex* $\mathbb{RP}^2$*-manifold.*
- A tiling of $\Omega = H^2$ in the projective model by triangles with angles $\pi/3, \pi/3, \pi/4$. The corresponding Coxeter group contains a finite index subgroup Γ such that Ω/Γ is a closed hyperbolic (and hence convex $\mathbb{RP}^2$-) surface.



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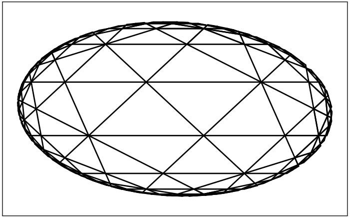

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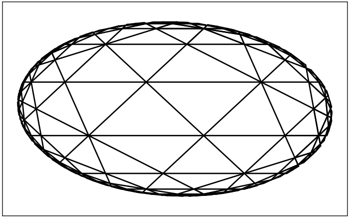

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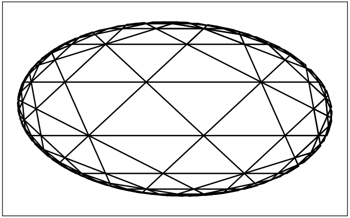

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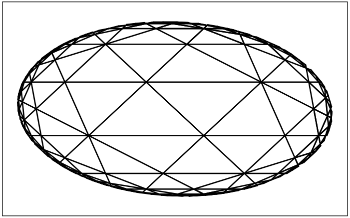

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- $\chi(\Sigma) < 0$: there will be other domains with fractal boundary determining convex $\mathbb{RP}^2$-structures M on Σ.
- (Kuiper 1954) $\partial\Omega$ is a conic and M is hyperbolic.
- (Benzécri 1960) $\partial\Omega$ is a C^1 convex curve.
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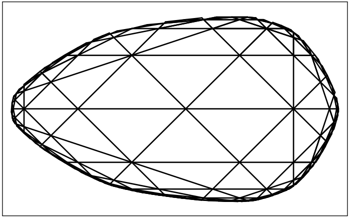

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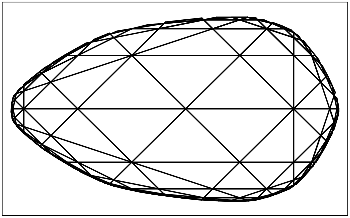

Convex $\mathbb{RP}^2$-structures

- $\chi(\Sigma) < 0$: there will be other domains with fractal boundary determining convex $\mathbb{RP}^2$-structures M on Σ.
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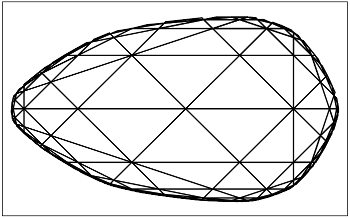

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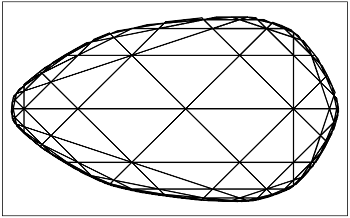

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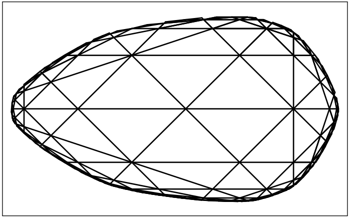

Globalizing the coordinate atlas

- Coordinate *changes* $g(C)$, for $C \subset U\alpha \cap U_\beta$, define fibration $E \xrightarrow{\Pi} M$, fiber X, structure group G;
- Product fibration over U_α:

$$E_\alpha := U_\alpha \times X \xrightarrow{\Pi_\alpha} U_\alpha :$$

- Since $C \longmapsto g(C) \in G$ is constant, the *foliations* of E_α defined by projections $E_\alpha \longrightarrow X$ define foliation $\mathfrak{F}$ of E;
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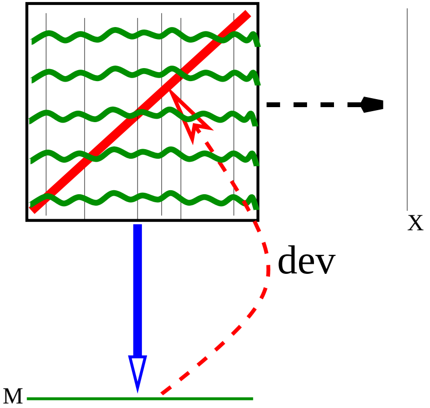
The *tangent* flat (G, X)-bundle



The developing section

- Graph the coordinate charts $U_\alpha \xrightarrow{\psi_\alpha} X$ to obtain sections of $E_\alpha := \Pi^{-1}(U_\alpha) = U_\alpha \times X$:

$$U_\alpha \xrightarrow{\mathsf{dev}_\alpha} U_\alpha \times X$$

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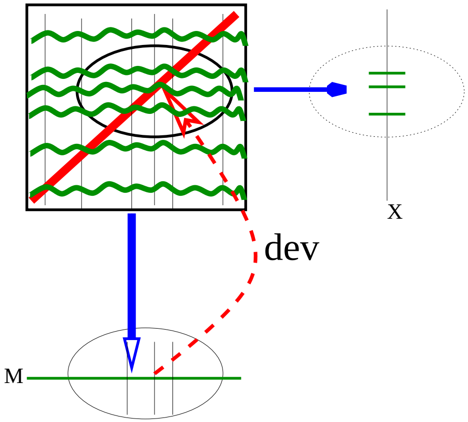

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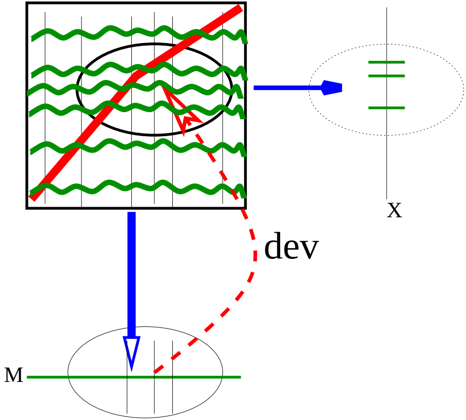

Development, holonomy

- Let $\widetilde{M} \longrightarrow M$ be a universal covering with deck group $\pi_1(M)$.
- This structure $(E, \mathfrak{F})$ is equivalent to a representation

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- $\widetilde{E} = \widetilde{M} \times X$, with $\pi_1(M)$-action defined by deck transformations on $\widetilde{M}$ and by ρ on G.
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The Ehresmann-Thurston Theorem

- Assume Σ *compact.* Two *nearby* structures with same holonomy are *isotopic:* equivalent by a diffeo in $\mathrm{Diff}(M)^0$,
- The holonomy representation ρ of a (G, X)-manifold M has an open neighborhood in $\mathrm{Hom}(\pi_1, G)$ of holonomy representations of *nearby* (G, X)-manifolds.


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Modeling structures on representations of π_1

- A *marked* (G, X)-*structure* on Σ is a diffeomorphism $\Sigma \xrightarrow{f} M$ where M is a (G, X)-manifold.
- Marked (G, X)-structures (f_i, M_i) are *isotopic* $\Longleftrightarrow \exists$ isomorphism $M_1 \xrightarrow{\phi} M_2$ with $\phi \circ f_1 \simeq f_2$.
- Holonomy defines a *local homeomorphism*

$$\mathfrak{D}_{(G,X)}(\Sigma) := \left\{ \text{Marked } (G, X)\text{-structures on } \Sigma \right\} / \text{Isotopy}$$

$$\xrightarrow{\text{hol}} \operatorname{Hom}(\pi_1(\Sigma), G)/G$$

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Change the marking!

- Let $\Sigma \xrightarrow{f} M$ be a marked (G, X)-structure.
- $\eta \in \mathsf{Diff}(\Sigma)$ acts: $(f, M) \longmapsto (f \circ \eta, M)$.
- *Mapping class group*

$$\mathsf{Mod}(\Sigma) := \pi_0\big(\mathsf{Diff}(\Sigma)\big)$$

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$$\mathsf{Mod}(\Sigma) \longrightarrow \mathsf{Out}\big(\pi_1(\Sigma)\big) := \mathsf{Aut}\big(\pi_1(\Sigma)\big)/\mathsf{Inn}\big(\pi_1(\Sigma)\big)$$

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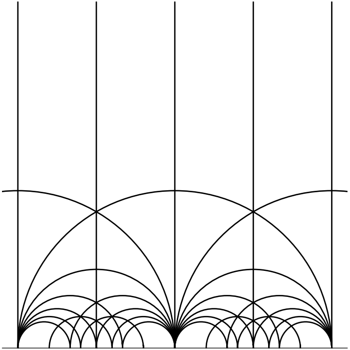
Marked Euclidean structures on the T^2

- *Euclidean geometry:* $X = \mathbb{R}^2$ and $G = \mathrm{Isom}(X)$
  $\mathfrak{D}_{(G,X)}(\Sigma)$ identifies with the upper half-plane H^2:
- Point $\tau \in \mathsf{H}^2 \longleftrightarrow$ Euclidean manifold $\mathbb{C}/\langle 1, \tau\rangle$.
- The marking is the choice of basis $1, \tau$ for $\pi_1(M)$.
- Changing the marking is the usual action of $\mathrm{PGL}(2, \mathbb{Z})$ on H^2 by linear fractional transformations which is properly discrete!.



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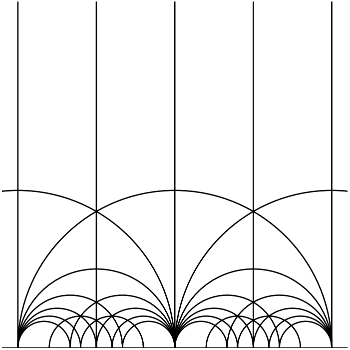

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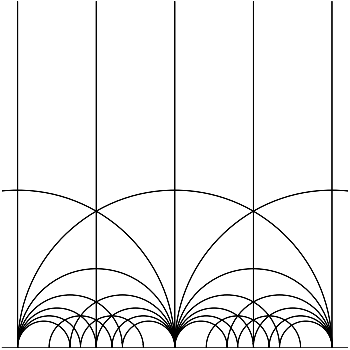

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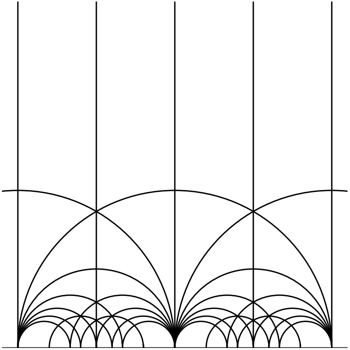

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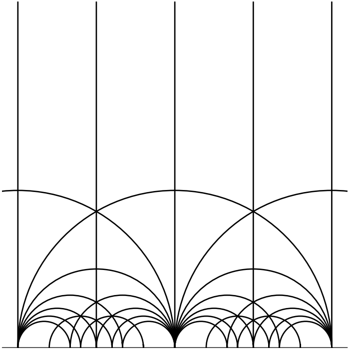

Complete affine structures on the 2-torus

- Kuiper (1954) Every complete affine closed orientable 2-manifold is equivalent to either:
  - *Euclidean:* $\mathbb{R}^2/\Lambda$, where Λ is a lattice of translations (all are affinely equivalent);
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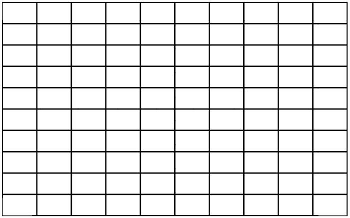

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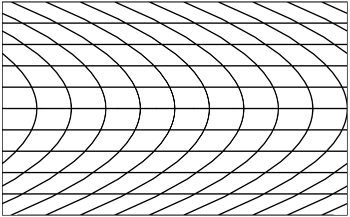

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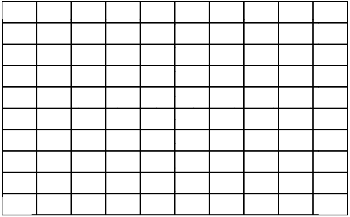

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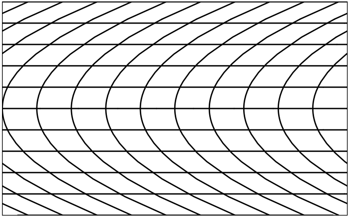

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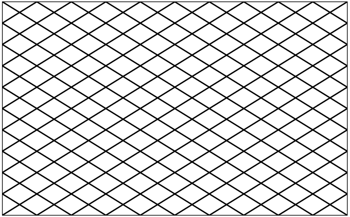

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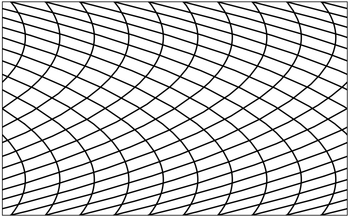

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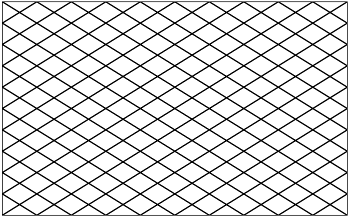

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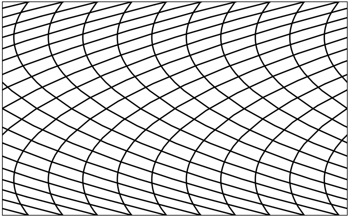

Chaotic dynamics on the deformation space

- Usually $\mathrm{Mod}(\Sigma)$ too dynamicly interesting to form a quotient.
- (Baues 2000) Deformation space homeomorphic $\mathbb{R}^2$, where origin $\{(0,0)\}$ corresponds to Euclidean structure;
- Mapping class group action is *linear* $\mathrm{GL}(2,\mathbb{Z})$-action on $\mathbb{R}^2$ — chaotic!
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Fricke spaces of hyperbolic structures

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Representations and their symmetries

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and G Lie group:.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

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admits an action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$:

$$\pi \xrightarrow{\phi} \pi \xrightarrow{\rho} G \xrightarrow{\alpha} G$$

where $(\phi, \alpha) \in \mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$, $\rho \in \mathrm{Hom}(\pi, G)$.
- The quotient

$$\mathrm{Hom}(\pi, G)/G \ := \ \mathrm{Hom}(\pi, G)/\big(\{1\} \times \mathrm{Inn}(G)\big)$$

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Algebraic structure of representation spaces

- G: *algebraic* Lie group.
- $\rho \longmapsto (\rho(X_1) \dots \rho(X_n))$ *embeds* $\mathrm{Hom}(\pi, G)$ onto an *algebraic subset* of G^n.
- Algebraic structure is $\{X_1, \dots, X_n\}$-independent and $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$-invariant.
- Geometric Invariant Theory quotient $\mathrm{Hom}(\pi, G)//G$ is $\mathrm{Out}(\pi)$-invariant.
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Character functions f_α on representation varieties

- Invariant function $G \xrightarrow{f} \mathbb{R} \Longrightarrow$ *Function* f_α on $\mathrm{Hom}(\pi, G)/G$

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  Conjugacy class of $\alpha \in \pi$ corresponds to free homotopy class of closed oriented loop $\alpha \subset \Sigma$.
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- Example: *Trace* $\mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{tr}} \mathbb{R}$
- Another example: *Displacement length* on $\mathrm{SL}(2, \mathbb{R})$:

$$\ell(A) := \min_{x \in \mathrm{H}^2} d\big(x, A(x)\big)$$

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Invariant functions in $\mathsf{PGL}(3,\mathbb{R}) \cong \mathsf{SL}(3,\mathbb{R})$

- Restrict to the subset $\mathsf{Hyp}_+ \subset \mathsf{SL}(3,\mathbb{R})$ consisting of *positive hyperbolic* elements (diagonalizable over $\mathbb{R}$):

$$A \sim \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

  where $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_1\lambda_2\lambda_3 = 1$.
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Marked length spectra in $\mathfrak{F}(\Sigma)$ and $\mathfrak{C}(\Sigma)$

- On $\mathfrak{F}(\Sigma)$, ℓ_α associates to a marked hyperbolic surface $\Sigma \approx M$ length of the unique closed geodesic homotopic to α in M.
- On $\mathfrak{C}(\Sigma)$, ℓ_α associates to a marked convex $\mathbb{RP}^2$-surface $\Sigma \approx M$ the Hilbert length of the unique closed geodesic homotopic to α in M.
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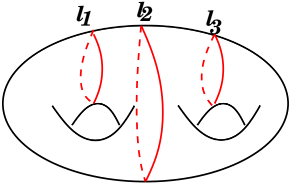
Fenchel-Nielsen coordinates on the Fricke space $\mathfrak{F}(\Sigma)$

- Cut Σ along N simple closed curves σ_i into 3-holed spheres (pants). $\Longrightarrow$ Explicit parametrization $\mathfrak{F}(\Sigma) \longrightarrow \mathbb{R}^{6g-6}$.



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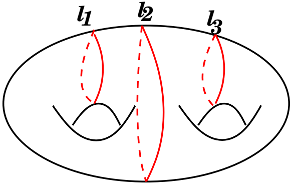

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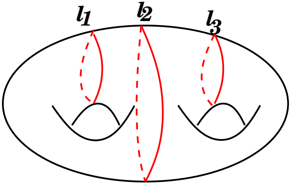

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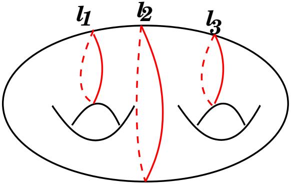

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Hyperbolic structures on three-holed spheres

- Let l_i be the length of the geodesic corresponding to σ_i. The hyperbolic structure on P_j is completely determined by the the three lengths of the components of ∂P_j.
- these length functions define a surjection

$$\mathfrak{F}(\Sigma) \xrightarrow{\ell} \mathbb{R}_+^N. \tag{1}$$

  which describes the hyperbolic structure on $M|\sigma$.
- The components of $\partial(M|\sigma)$ are identified $\sigma_i^- \longleftrightarrow \sigma_i^+$, one pair for each component $\sigma_i \subset \sigma$.
- For each σ_i, choose $\tau_i \in \mathbb{R}$ and reidentify $M|\sigma$ $\sigma_i^- \longleftrightarrow \sigma_i^+$, one pair for each σ_i, obtaining a new marked hyperbolic surface

$$S \approx M_{\tau_1,\dots,\tau_N}$$

  (Fenchel-Nielsen twists, Thurston earthquakes).

Hyperbolic structures on three-holed spheres

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Fenchel-Nielsen twists (earthquakes)

- Defines an $\mathbb{R}^N$-action which is simply transitive on the fibers of $\mathfrak{F}(\Sigma) \xrightarrow{\ell} \mathbb{R}^N_+$.
- (Wolpert 1977) The symplectic form is

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- *Completely integrable Hamiltonian system:* ℓ is a Cartesian projection for *symplectomorphism*.

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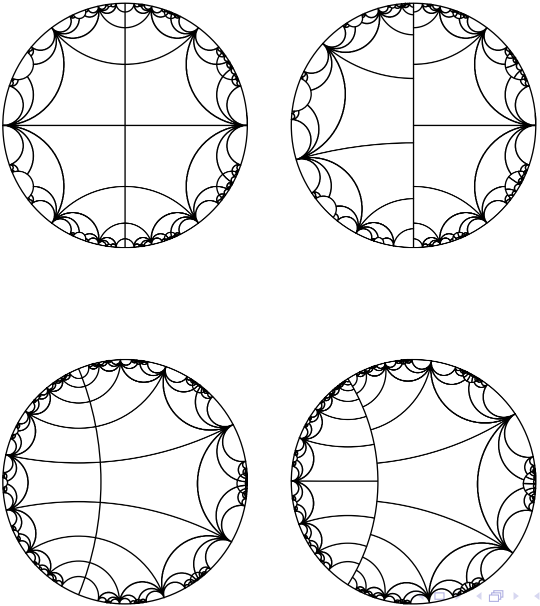
Some earthquake deformations in the universal covering



Geometry of $\mathfrak{C}(\Sigma)$

- Hong Chan Kim (1999) generalized Wolpert's theorem to define a a *symplectomorphism*

$$\mathfrak{C}(\Sigma) \longrightarrow \mathbb{R}^{16g-6}$$

- $\exists$ *natural* completely integrable system in this case?
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- Σ *oriented* closed surface and $\mathbb{B}$ Ad-invariant nondegeneate symmetric pairing on $\mathfrak{g}$.
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- This pairing is *skew-symmetric*, and hence defines an exterior 2-form on the smooth part of $\mathrm{Hom}(\pi, G)/G$.
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Hamiltonian twist flows on Hom(π, G)

- The Hamiltonian vector field Ham(f_α) associated to f and α assigns to a representation ρ in Hom(π, G) a tangent vector

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- It is represented by the (Poincaré dual) cycle-with-coefficient supported on α and with coefficient

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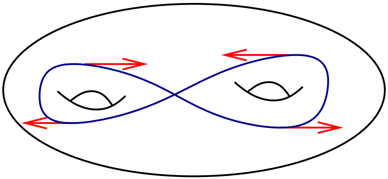

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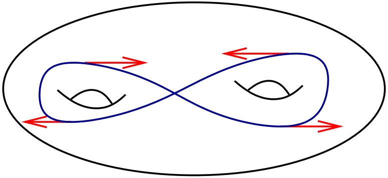

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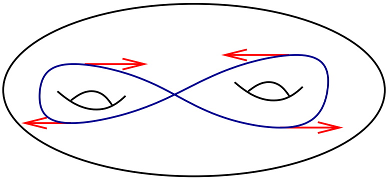

The one-parameter subgroup associated to an invariant function

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$$G \xrightarrow{f} \mathbb{R}$$
and $A \in G \implies$ one-parameter subgroup
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where $F(A) \in \mathfrak{g}$.
- Centralizes A:
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- When α is a simple closed curve, then a flow Φ_t on $\mathrm{Hom}(\pi, G)$ exists, which covers the (local) flow of the Hamiltonian vector field $\mathrm{Ham}(f_\alpha)$.
- When α is, for example, the nonseparating curve A_1 in the standard presentation

$$\pi = \langle A_1, B_1, \dots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots, A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$$

this flow has the following description in terms of generators:.

- $\Phi_t(\gamma) = \rho(\gamma)$ is constant if γ is either A_i for $1 \le i \le g$ or B_i for $2 \le i \le g$.
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Twist and bulging deformations for $\mathbb{RP}^2$-structures

- Apply the previous general construction to $G = \mathrm{SL}(3,\mathbb{R})$ and the two invariant functions ℓ, β defined earlier:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \xrightarrow{(\ell,\beta)} \begin{pmatrix} \log(\lambda_1) - \log(\lambda_3) \\ \log(\lambda_2) \end{pmatrix}$$

- The corresponding one-parameter subgroups in $\mathrm{PGL}(3,\mathbb{R})$ are:

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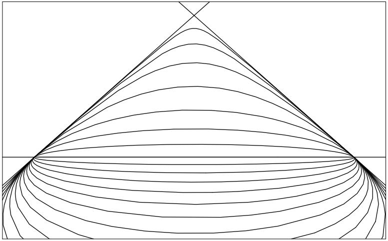
Bulging conics along a triangle in $\mathbb{RP}^2$

- When applied to a hyperbolic structure, the flow of $\text{Ham}(\ell_\alpha)$ is just the ordinary Fenchel-Nielsen earthquake deformation and the developing image Ω is unchanged.
- However, the flow of $\text{Ham}(\beta_\alpha)$ changes Ω by *bulging* it along a triangle tangent to $\partial\Omega$.



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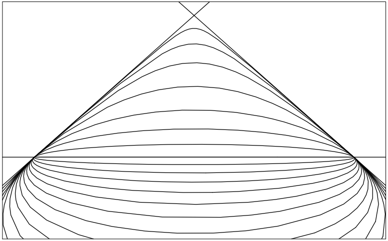

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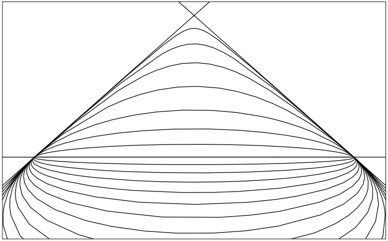

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- Start with a properly domain Ω whose boundary $\partial\Omega$ is strictly convex and C^1. (For example, $\partial\Omega$ a conic.) Each geodesic embeds in a triangle tangent to $\partial\Omega$.
- Choose a collection Λ of disjoint lines in Ω, with instructions how to deform along Λ (for each $\lambda \in \Lambda$, a one-parameter subgroup of $\mathrm{SL}(3, \mathbb{R})$ preserving λ.
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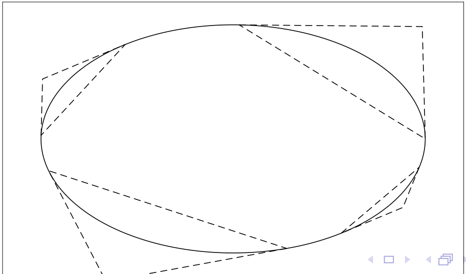

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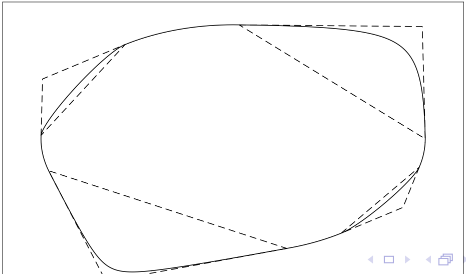
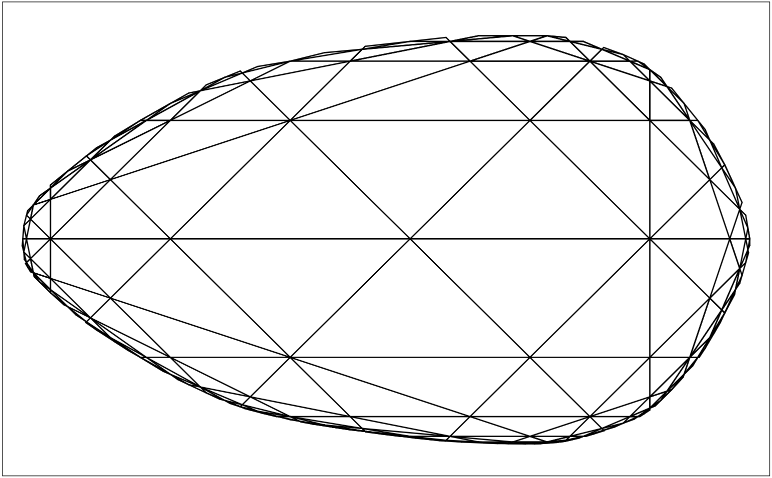


A domain in $\mathbb{RP}^2$ covering a closed surface



Iterated bulging of convex domains in $\mathbb{RP}^2$: Speculation

- If Ω covers a closed convex $\mathbb{RP}^2$-surface with $\chi < 0$, then $\partial\Omega$ is obtained from a conic by iterated bulgings and earthquakes.
- Is every properly convex domain $\Omega \subset \mathbb{RP}^2$ with strictly convex C^1 boundary obtained by iterated bulging-earthquaking?
- Thurston proved that any two marked hyperbolic structures on Σ can be related by (left)-earthquake along a unique *measured geodesic lamination.* Generalize this to convex $\mathbb{RP}^2$-structures.

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